Aequationes Math. 57 (1999) 87–107
0001-9054/99/010087-21 $ 1.50+0.20/0



Aequationes Mathematicae

# Invariant and complementary quasi-arithmetic means

Janusz Matkowski

**Summary.** Let $I \subset \mathbb{R}$ be an interval. Under the assumptions that $\phi,\ \psi,\ \gamma : I \to \mathbb{R}$ are one-to-one, $\gamma(I)$ is an interval, and at least one of the functions $\phi \circ \gamma^{-1},\ \psi \circ \gamma^{-1}$ is twice continuously differentiable on $\gamma(I)$, we determine all the quasi-arithmetic means $M_\gamma,\ M_\phi,\ M_\psi$ satisfying the functional equation $M_\gamma(M_\phi(x,y),\ M_\psi(x,y)) = M_\gamma(x,y)$ which can be interpreted in the following two ways: the mean $M_\gamma$ is invariant with respect to $M_\phi$ and $M_\psi$, or $M_\psi$ is "complementary" to $M_\phi$ with respect to the mean $M_\gamma$.

**Mathematics Subject Classification (1991).** 39B22, 39B12, 26A18.

**Keywords.** Means, invariant mean, complementary means with respect to a given mean, quasi-arithmetic means, positively homogeneous mean, composition of means, functional equations.

## Introduction

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to I$ is said to be a *mean* on $I^2$ if for all $x,\ y \in I$,

$$\min(x,y) \le M(x,y) \le \max(x,y).$$

A mean $M$ on $I^2$ is called *strict* if

$$x,\ y \in I, \qquad x \ne y \iff \min(x,y) < M(x,y) < \max(x,y),$$

and *symmetric* if $M(x,y) = M(y,x)$ for all $x,\ y \in I$. If $I = (0,\infty)$ and

$$M(tx,ty) = tM(x,y), \qquad t,\ x,\ y > 0,$$

then the mean $M$ is called *positively homogeneous*. Let $M,\ N : I^2 \to I$ be given means. A mean $K : I^2 \to I$ is called $(M,N)$-*invariant*, if

$$K(M(x,y),N(x,y)) = K(x,y), \qquad x,\ y \in I.$$

We show that if the means $M$ and $N$ are continuous, and at least one of them is strict then there exists a unique $(M,N)$-invariant mean (cf. also [9] where a more general case is considered).

Let $K : I^2 \to I$ be a fixed symmetric mean which is continuous and strictly increasing with respect to each variable. In Section 1 we show that for every mean $M : I^2 \to I$ there exists a unique mean $M^{(K)} : I^2 \to I$ satisfying the equation

$$K(M(x,y), M^{(K)}(x,y)) = K(x,y), \qquad x,\, y \in I.$$

The mean $M^{(K)}$ is called the *$K$-complementary mean to $M$*. Note that the mean $K$ is $(M, M^{(K)})$-invariant. We present some general properties of the operation $M \to M^{(K)}$; in particular, it turns out to be an involution, i.e. $(M^{(K)})^{(K)} = M$.

One of the most important classes of symmetric means is the family of quasi-arithmetic means. Recall that a mean $M : I^2 \to I$ is called *quasi-arithmetic* if there is a continuous one-to-one function $\phi : I \to \mathbb{R}$ (called a *generator* of $M$) such that $M = M_\phi$ where

$$M_\phi(x,y) := \phi^{-1}\Big(\frac{\phi(x) + \phi(y)}{2}\Big), \qquad x,\, y \in I.$$

If $M$ and $N$ are quasi-arithmetic means then, obviously, the unique $(M, N)$-invariant mean need not be quasi-arithmetic. In particular, if the means $K$ and $M$ are quasi-arithmetic, then the $K$-complementary mean to $M$ need not be of the same type.

In this connection the following problem arises: characterize the quasi-arithmetic means $M_\phi$, $M_\psi$, $M_\gamma$ such that $M_\gamma$ is $(M_\phi,\, M_\psi)$-invariant. We solve it assuming that at least one of the functions $\phi \circ \gamma^{-1}$ and $\psi \circ \gamma^{-1}$ is twice continuously differentiable. Note that this problem is actually equivalent to the following one. Given a quasi-arithmetic mean $K$, determine all quasi-arithmetic means $M$ such that $M^{(K)}$ is also quasi-arithmetic. The key results are contained in Section 2, where we assume that $K$ is the arithmetic mean. In Section 3, applying the results of Section 2, we solve the problem for an arbitrary quasi-arithmetic mean $K$. Some open problems are proposed.

## 1. Complementary means

We start with the following

**Remark 1.** Let $I \subset \mathbb{R}$ be an interval and $K : I^2 \to I$ a fixed symmetric mean which is continuous and strictly increasing with respect to the first variable. Then for every mean $M : I^2 \to I$ there exists a unique function $N : I^2 \to I$ such that

$$K(M(x,y), N(x,y)) = K(x,y), \qquad x,\, y \in I,$$

and $N$ is a mean on $I^2$.

*Proof.* Take arbitrary $x, y \in I$, $x \le y$, and define the function $f : [x,y] \to I$ by

$$f(z) := K(M(x,y), z), \qquad z \in [x,y].$$

The symmetry of $K$, the increasing monotonicity of $K$ with respect to the first variable, and the inequality $M(x,y) \le y$ imply that

$$f(x) = K(M(x,y), x) = K(x, M(x,y)) \le K(x,y).$$

The symmetry of $K$ implies that it is increasing with respect to the second variable. Now the inequality $x \le M(x,y)$ gives

$$f(y) = K(M(x,y), y) \ge K(x,y).$$

Since $f$ is continuous and strictly increasing in the interval $[x,y]$, there is a unique $z = N(x,y) \in [x,y]$ such that $f(z) = K(x,y)$. This completes the proof. □

The mean $N$ is called *complementary to $M$ with respect to $K$* (shortly, *$K$-complementary mean to $M$*) and we write

$$M^{(K)} := N.$$

If $K$ is quasi-arithmetic then, for an arbitrary mean $M$, the $K$-complementary mean to $M$ is easy to determine. Namely, we have the following

**Remark 2.** Let $I \subset \mathbb{R}$ be an interval, $\phi : I \to \mathbb{R}$ continuous strictly monotonic, and $M : I^2 \to \mathbb{R}$ a mean. Then $N : I^2 \to I$ given by

$$N(x,y) := \phi^{-1}(\phi(x) + \phi(y) - \phi(M(x,y)), \qquad x,\, y \in I,$$

is the $M_\phi$-complementary mean to $M$; moreover $M_\phi$ is $(M,N)$-invariant.

In the sequel $A$, $G$, $H$ stand for the arithmetic, geometric and harmonic means, respectively.

Let us note some general properties of $K$-complementary means.

**Proposition 1.** *Suppose that $I \subset \mathbb{R}$ is an interval and $K : I^2 \to I$ a fixed symmetric mean which is continuous and strictly increasing with respect to the first variable. Let $M : I^2 \to I$ be an arbitrary mean. Then*

$\mathbf{1^0}$ $(M^{(K)})^{(K)} = M$ *(i.e. the mapping $M \to M^{(K)}$ is involutory).*

$\mathbf{2^0}$ *A mean $M$ is $K$-self-complementary, i.e. $M^{(K)} = M$, iff $M = K$.*

$\mathbf{3^0}$ *$M$ is symmetric iff $M^{(K)}$ is symmetric.*

**$4^0$** *Suppose that $K$, $M$ and $M^{(K)}$ are quasi-arithmetic means on $I$, i.e. there are one-to-one and continuous $\gamma$, $\phi$, $\psi : I \to \mathbb{R}$ such that*

$$K = M_\gamma, \qquad M = M_\phi, \qquad M^{(K)} = M_\psi.$$

*If $\gamma$, $\phi^{-1}$, $\psi^{-1}$ are convex (concave) and $\gamma$ is increasing (decreasing) respectively on $\phi(I)$ and $\psi(I)$, then*

$$K = M = M^{(K)} = A.$$

**$5^0$** *Suppose that $I = (0, \infty)$. If $K$, $M$ and $M^{(K)}$ are quasi-arithmetic, and $K$ and $M$ are positively homogeneous, then one of the following cases occurs:*

$$K = M = M^{(K)}; \;\; K = G, \;\; M = A, \;\; M^{(K)} = H; \;\; K = G, \;\; M = H, \;\; M^{(K)} = A.$$

**$6^0$** *$K$ is $(M, M^{(K)})$-invariant.*

*Proof.* We omit the easy arguments for $1^0 - 3^0$ and $6^0$. To prove $4^0$ note that

$$M_\gamma(M_\phi(x, y), M_\psi(x, y)) = M_\gamma(x, y), \qquad x, y \in I,$$

which by the definition of the quasi-arithmetic mean can be written in the form of the functional equation

$$\gamma\Big(\phi^{-1}\Big(\frac{\phi(x) + \phi(y)}{2}\Big)\Big) + \gamma\Big(\psi^{-1}\Big(\frac{\psi(x) + \psi(y)}{2}\Big)\Big) = \gamma(x) + \gamma(y), \qquad x, y \in I.$$

Assume that $\phi^{-1}$ and $\psi^{-1}$ are convex and $\gamma$ is increasing. Then

$$\gamma\Big(\phi^{-1}\Big(\frac{\phi(x) + \phi(y)}{2}\Big)\Big) \le \gamma\Big(\frac{x + y}{2}\Big), \qquad x, y \in I,$$

$$\gamma\Big(\psi^{-1}\Big(\frac{\psi(x) + \psi(y)}{2}\Big)\Big) \le \gamma\Big(\frac{x + y}{2}\Big), \qquad x, y \in I,$$

and, consequently,

$$2\gamma\Big(\frac{x + y}{2}\Big) \ge \gamma(x) + \gamma(y), \qquad x, y \in I,$$

which means that $\gamma$ is concave on $I^2$. Since, by assumption, $\gamma$ is convex, it must be affine, and our equation gives the relation

$$\phi^{-1}\Big(\frac{\phi(x) + \phi(y)}{2}\Big) + \psi^{-1}\Big(\frac{\psi(x) + \psi(y)}{2}\Big) = x + y, \qquad x, y \in I.$$

Since, by the convexity of $\phi^{-1}$ and $\psi^{-1}$,

$$\phi^{-1}\Big(\frac{\phi(x)+\phi(y)}{2}\Big) \le \frac{x+y}{2}, \qquad \psi^{-1}\Big(\frac{\psi(x)+\psi(y)}{2}\Big) \le \frac{x+y}{2}, \qquad x,\, y \in I,$$

the above relation implies that

$$\phi^{-1}\Big(\frac{\phi(x)+\phi(y)}{2}\Big) = \frac{x+y}{2}, \qquad \psi^{-1}\Big(\frac{\psi(x)+\psi(y)}{2}\Big) = \frac{x+y}{2}, \qquad x,\, y \in I.$$

Thus $\phi$ and $\psi$ are also affine and, consequently, the functions $K$, $M$ and $M^{(K)}$ are the arithmetic means. In the remaining cases the arguments are similar.

The statement $5^0$ is a consequence of the main result of Kahlig and Matkowski [5].

Let us note that Proposition $1.5^0$ gives a full characterization of all positively homogeneous quasi-arithmetic means $K$ and $M$ such that $M^{(K)}$ is also quasi-arithmetic. We shall see in the sequel that dropping the assumption of positive homogeneity makes the problem of characterization essentially more difficult.

## 2. Complementary quasi-arithmetic means with respect to the arithmetic mean

In this section we deal with the following problem. Let $I \subset \mathbb{R}$ be an interval. Determine all quasi-arithmetic means $M_\phi$ on $I^2$ such that $(M_\phi)^{(A)}$, the $A$-complementary mean of $M_\phi$, is also quasi-arithmetic, i.e. there is a (continuous one-to-one) function $\psi : I \to \mathbb{R}$ such that

$$M_\psi = (M_\phi)^{(A)}.$$

According to the definition of the $A$-complementarity this problem reduces to the following one. Determine all continuous and strictly monotonic solutions $\phi$, $\psi : I \to \mathbb{R}$ of the functional equation

$$A(M_\phi(x,y), M_\psi(x,y)) = A(x,y), \qquad x, y \in I,$$

which is equivalent to the $(M_\phi, M_\psi)$-invariance of the arithmetic mean and, obviously, can be written in the more explicit form

$$\phi^{-1}\Big(\frac{\phi(x)+\phi(y)}{2}\Big) + \psi^{-1}\Big(\frac{\psi(x)+\psi(y)}{2}\Big) = x+y, \qquad x,\, y \in I. \tag{1}$$

In this section we solve this functional equation assuming that $\phi$ and $\psi$ are twice continouosly differentiable.

We begin with some auxiliary results.

**Lemma 1.** *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\phi$, $\psi : I \to \mathbb{R}$ are one-to-one and twice differentiable functions satisfying equation* (1)*. Then there exists $c \in \mathbb{R}$, $c \neq 0$, such that*

$$\phi'(x)\psi'(x) = c, \qquad x \in I.$$

*Proof.* Denote by $Z$ the set of all $x \in I$ such that $\phi'(x) = 0$ or $\psi'(x) = 0$. The continuity of $\phi'$ and $\psi'$, and the strict monotonicity of $\phi$ and $\psi$ imply that $Z$ is closed in $I$ and $\text{int}(Z) = \emptyset$ . Let $J$ be a maximal interval such that $J \subset I \setminus Z$. As $M_\phi$ and $M_\psi$ are means, we have

$$\phi^{-1}(\frac{\phi(x) + \phi(y)}{2}) + \psi^{-1}(\frac{\psi(x) + \psi(y)}{2}) = x + y, \qquad x,\, y \in J.$$

Differentiation of both sides with respect to $x$ and $y$ gives

$$\frac{\phi'(x)}{\phi'(\phi^{-1}(\frac{\phi(x)+\phi(y)}{2}))} + \frac{\psi'(x)}{\psi'(\psi^{-1}(\frac{\psi(x)+\psi(y)}{2}))} = 2,$$

$$\frac{\phi'(y)}{\phi'(\phi^{-1}(\frac{\phi(x)+\phi(y)}{2}))} + \frac{\psi'(y)}{\psi'(\psi^{-1}(\frac{\psi(x)+\psi(y)}{2}))} = 2$$

for all $x$, $y \in J$. Subtracting these equations we obtain

$$\frac{\phi'(y) - \phi'(x)}{\phi'(\phi^{-1}(\frac{\phi(x)+\phi(y)}{2}))} + \frac{\psi'(y) - \psi'(x)}{\psi'(\psi^{-1}(\frac{\psi(x)+\psi(y)}{2}))} = 0, \qquad x,\, y \in J.$$

Dividing both sides by $y - x$ and then letting $y \to x$ we obtain

$$\frac{\phi''(x)}{\phi'(x)} + \frac{\psi''(x)}{\psi'(x)} = 0, \qquad x \in J.$$

It easily follows that there is a constant $c \in \mathbb{R}$, $c \neq 0$, such that $\phi'(x)\psi'(x) = c$ for all $x \in J$. The continuity of $\phi'$ and $\psi'$ implies that $J = I$, and the proof is completed. □

By $\mathbb{N}_0$ denote the set of all nonnegative integers, i.e. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In the sequel an important role is played by the following result on compositions of means.

**Lemma 2.** *Suppose that $M$, $N : I^2 \to I$ are continuous means, and at least one of them is strict. Let the functions $M_k$, $N_k : I^2 \to I$, $k \in \mathbb{N}_0$, be defined by*

$$M_0(x, y) := M(x, y), \qquad N_0(x, y) := N(x, y),$$

$$M_{k+1}(x,y) := M\left(M_k(x,y), N_k(x,y)\right), \qquad N_{k+1}(x,y) := N\left(M_k(x,y), N_k(x,y)\right).$$

*Then*

**$1^0$** *for every $k \in \mathbb{N}_0$, the functions $M_k$ and $N_k$ are continuous means on $I^2$;*

**$2^0$** *the sequences $(M_k)$ and $(N_k)$ converge on $I^2$ to the same function $K : I^2 \to I$, i.e.*

$$\lim_{k\to\infty} M_k(x,y) = K(x,y) = \lim_{k\to\infty} N_k(x,y), \qquad x, y \in I;$$

*moreover $K$ is a continuous and $(M,N)$-invariant mean on $I^2$;*

**$3^0$** *if $M$ and $N$ are strict means then so is $K$;*

**$4^0$** *if $I = (0,\infty)$ and $M$, $N$ are positively homogeneous, then $K$ is positively homogeneous.*

*Proof.* The part $1^0$ is obvious. To prove $2^0$ suppose that $M$ is a strict mean and define $m_k$, $n_k : I^2 \to I$, $k \in \mathbb{N}_0$, by

$$m_k(x,y) := \min\left(\left(M_k(x,y)\right), \left(N_k(x,y)\right)\right),$$

$$n_k(x,y) := \max\left(\left(M_k(x,y)\right), \left(N_k(x,y)\right)\right).$$

The functions $M_k$, $N_k$, $m_k$, $n_k$ are continuous and, as $M$ and $N$ are means, we have

$$m_0(x,y) \le m_1(x,y) \le \ldots \le m_k(x,y) \le n_k(x,y) \le \ldots \le n_1(x,y) \le n_0(x,y),$$

and

$$M_k(x,y), N_k(x,y) \in [m_k(x,y), n_k(x,y)] \tag{2}$$

for all $k \in \mathbb{N}_0$ and $x$, $y \in I$. It follows that the sequences $(m_k)$ and $(n_k)$ converge on $I^2$. Thus there exist $m_\infty$, $n_\infty : I^2 \to I$ such that

$$\lim_{k\to\infty} m_k(x,y) =: m_\infty(x,y) \le n_\infty(x,y) := \lim_{k\to\infty} n_k(x,y),$$

for all $x$, $y \in I$. Since the functions of both sequences are continuous, $(m_k)$ is increasing and $(n_k)$ is decreasing, the function $m_\infty$ is lower semicontinuous, and $n_\infty$ is upper semicontinuous on $I^2$. Suppose that there are $x$, $y \in I$ such that $m_\infty(x,y) < n_\infty(x,y)$. Hence, as $M$ is a strict mean, we would get

$$M\left(m_\infty(x,y), n_\infty(x,y)\right) \in \left(m_\infty(x,y), n_\infty(x,y)\right).$$

Now the continuity of $M$ implies that, for sufficiently large $k$,

$$M\left(M_k(x,y), N_k(x,y)\right) \in \left(m_\infty(x,y), n_\infty(x,y)\right),$$

i.e., for sufficiently large $k$,

$$M_{k+1}(x,y) \in (m_\infty(x,y), n_\infty(x,y)) .$$

Hence, by the definition of $(m_k)$, for sufficiently large $k$,

$$m_{k+1}(x,y) \in (m_\infty(x,y), n_\infty(x,y)) ,$$

which is a contradiction. This proves that for all $x,\ y \in I$,

$$m_\infty(x,y) = n_\infty(x,y).$$

Define $K : I^2 \to I$ by

$$K(x,y) := m_\infty(x,y), \qquad x, y \in I.$$

The function $K$, being lower and upper semicontinuous, is continuous. The pointwise convergence of the sequences $(M_k)$ and $(N_k)$ to $K$ is a consequence of relation (2). Take $x,\ y \in I$ and $x \neq y$. Without any loss of generality we can assume that $x \le y$. Then

$$x \le M(x,y) \le y, \qquad x \le N(x,y) \le y.$$

Since

$$\min (M(x,y), N(x,y)) \le K(x,y) \le \max (M(x,y), N(x,y)) ,$$

we infer that $K$ is a mean.

Now define the sequences $(\bar{M}_k)$, $(\bar{N}_k)$ inductively by

$$\bar{M}_0(x,y) := M(x,y), \qquad \bar{N}_0(x,y) := N(x,y),$$

$$\bar{M}_{k+1}(x,y) := \bar{M}_k (M(x,y), N(x,y)) , \qquad \bar{N}_{k+1}(x,y) := \bar{N}_{k+1} (M(x,y), N(x,y)) ,$$

$x,\ y \in I,\ k \in \mathbb{N}_0$. By an obvious induction we have $\bar{M}_k = M_k$ and $\bar{N}_k = N_k$ for all $k \in \mathbb{N}_0$, and, consequently,

$$\begin{aligned} K(x,y) &= \lim_{k\to\infty} M_k(x,y) := \lim_{k\to\infty} \bar{M}_{k+1}(x,y) = \lim_{k\to\infty} \bar{M}_k (M(x,y), N(x,y)) \\ &= K (M(x,y), N(x,y)) , \end{aligned}$$

for all $x,\ y \in I$, which proves that $K$ is $(M,N)$-invariant. This completes the proof of $2^0$. To prove $3^0$ take $x,\ y \in I,\ x < y$. $M$ and $N$ are strict means,

$$x < M(x,y) < y, \qquad x < N(x,y) < y,$$

hence

$$x < \min (M(x,y), N(x,y)) < K(x,y) < \max (M(x,y), N(x,y)) < y.$$

Now the definition of $K$ implies that it is a strict mean. Since $4^0$ is obvious, the proof is completed. □

Lemma 2 improves a result of Matkowski–Wróbel [7] (cf. also Páles [10]) where both $M$ and $N$ are assumed to be strict.

**Remark 3.** Let $M$, $N : I^2 \to I$ be continuous means such that at least one of them is strict. Let $M_k$ and $N_k$, $k \in \mathbb{N}_0$, be defined as in Lemma 2. Define a mapping $T : I^2 \to I^2$ by $T := (M, N)$. It is easy to verify that $T^k$, the $k$-th iterate of $T$, is given by $T^k = (M_k, N_k)$. Therefore, in view of Lemma 2, we have

$$\lim_{k\to\infty} T^k = (K, K),$$

where $K$ is given in Lemma 2. A finite dimensional counterpart of this lemma is given in [9] (cf. also Flor and Halter-Koch [3] where a more special class of mappings is considered).

**Lemma 3.** *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\phi$, $\psi : I \to \mathbb{R}$ are continuous strictly monotonic functions such that equation* (1) *is satisfied. If $M$, $N : I^2 \to I$ are given by*

$$M := M_\phi, \qquad N := M_\psi,$$

*then the sequences $(M_k)$ and $(N_k)$ of means defined in Lemma 2 converge to the arithmetic mean on $I^2$.*

*Proof.* For all $k \in \mathbb{N}_0$ we have

$$M_k(x, y) + N_k(x, y) = x + y, \qquad x,\, y \in I. \tag{3}$$

In fact, by (1) and the definitions of $M_0$ and $N_0$, this relation is obviously true for $k = 0$. Suppose (3) holds true for a nonnegative integer $k$. Hence, making use of the definition of $M_k$ and $N_k$, and (1) we get

$$\begin{aligned} M_{k+1}(x, y) + N_{k+1}(x, y) &= M\left(M_k(x, y), N_k(x, y)\right) + N\left(M_k(x, y), N_k(x, y)\right) \\ &= M_k(x, y) + N_k(x, y) = x + y \end{aligned}$$

for all $x$, $y \in I$, and the induction proves (3).

In view of Lemma 2 there exists a mean $K : I^2 \to I$ such that

$$\lim_{k\to\infty} M_k(x, y) = \lim_{k\to\infty} N_k(x, y) = K(x, y), \qquad x,\, y \in I.$$

Now letting $k \to \infty$ in (3) gives $2K(x, y) = x + y$ for all $x$, $y \in I$, which was to be shown. □

**Remark 4.** A function $\phi : I \to \mathbb{R}$ is said to be Jensen affine if

$$\phi\left(\frac{x+y}{2}\right) = \frac{\phi(x)+\phi(y)}{2}, \qquad x,\, y \in I.$$

It is well known (Kuczma [6], p. 315, Theorem 1) that $\phi$ is Jensen affine on $I$ if, and only if, there exist an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$\phi(x) = \alpha(x) + c, \qquad x \in I.$$

If moreover $\phi$ is continuous at least at one point (or bounded on one side on a set of positive Lebesgue measure) then there is $a \in \mathbb{R}$ such that (cf. Aczél [1], p. 43)

$$\phi(x) = ax + c, \qquad x \in I.$$

Now we can prove the main result of this section.

**Theorem 1.** *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\phi$, $\psi : I \to \mathbb{R}$ are one-to-one and at least one of them is twice continuously differentiable. Then $\phi$ and $\psi$ satisfy the functional equation*

$$\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x + y, \qquad x,\, y \in I, \tag{1}$$

*if, and only if, one of the following two cases occurs:*

$1^0$ *there exist $a_1$, $a_2 \in \mathbb{R} \setminus \{0\}$, $b_1$, $b_2 \in \mathbb{R}$, such that*

$$\phi(x) = a_1 x + b_1, \qquad \psi(x) = a_2 x + b_2, \qquad x \in I;$$

$2^0$ *there exist $a$, $c_1$, $c_2 \in \mathbb{R} \setminus \{0\}$, $b_1$, $b_2 \in \mathbb{R}$, such that*

$$\phi(x) = c_1 e^{ax} + b_1, \qquad \psi(x) = c_2 e^{-ax} + b_2, \qquad x \in I.$$

*Proof.* Suppose that, for instance, $\phi$ is twice continuously differentiable on $I$. Since equation (1) can be written in the form

$$\psi(x) = 2\psi\left(x + y - \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right) - \psi(y), \quad x, y \in I,$$

where the function

$$(x, y) \longrightarrow x + y - \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right), \quad x, y \in I,$$

is twice differentiable, Járai's regularity theorem [4] implies that $\psi$ is twice continuously differentiable.

In view of Lemma 1 we have $\phi'(x) \neq 0 \neq \psi'(x)$ for all $x \in I$. Differentiation of both sides of (1) with respect to $x$ gives

$$\frac{\phi'(x)}{\phi'\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right)} + \frac{\psi'(x)}{\psi'\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)} = 2$$

for all $x$, $y \in I$. By Lemma 1 there exists $c \in \mathbb{R}$, $c \neq 0$, such that

$$\phi'(x)\psi'(x) = c, \qquad x,\, y \in I.$$

It follows that

$$\frac{\phi'(x)}{\phi'\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right)} + \frac{\phi'\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)}{\phi'(x)} = 2, \qquad x, y \in I.$$

As equation (1) is symmetric with respect to $x$ and $y$, we also have

$$\frac{\phi'(y)}{\phi'\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right)} + \frac{\phi'\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)}{\phi'(y)} = 2, \qquad x, y \in I.$$

Subtracting these two equations by sides we obtain,

$$\frac{\phi'(x)-\phi'(y)}{\phi'\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right)} + \frac{\phi'\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)}{\phi'(x)\phi'(y)}\left(\phi'(y)-\phi'(x)\right) = 0,$$

and consequently, for all $x$, $y \in I$,

$$\begin{aligned}&(\phi'(y)-\phi'(x))\\ &\times\left\{\phi'\left[\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right]\phi'\left[\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right] - \phi'(x)\phi'(y)\right\} = 0.\end{aligned}$$

Suppose that there is a subinterval $J$ such that $\phi'(y) - \phi'(x) = 0$ for all $x$, $y \in J \subset I$, i.e. $\phi(x) = a_1 x + b_1$, $x \in J$, for some $a_1$, $b_1 \in \mathbb{R}$. Making use of (1) we get $\psi(x) = a_2 x + b_2$, $x \in J$, for some $a_2$, $b_2 \in \mathbb{R}$, and it is easy to see that, in the above equation, the second factor also vanishes for all $x$, $y \in J$. Therefore the functions $\phi$ and $\psi$ statisfy the equation

$$\phi'\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right)\phi'\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right) = \phi'(x)\phi'(y), \qquad x, y \in I.$$

Since $\phi' \neq 0$ on $I$, the Darboux property of the derivative implies that $\phi'$ is either positive or negative. Of course, without any loss of generality, we can assume that $\phi' > 0$ on $I$. Hence the function

$$\gamma := \log \circ \phi'$$

satisfies the functional equation

$$\gamma\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right)+\gamma\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)=\gamma(x)+\gamma(y), \qquad x,\, y \in I.$$

Put for all $x,\, y \in I$

$$M_0(x,y) := \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right), \qquad N_0(x,y) := \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right),$$

and define the sequences of means $(M_k)$, $(N_k)$ as in Lemma 2. In view of this lemma there exists a strict mean $K$ on $I^2$ such that

$$\lim_{k\to\infty} M_k(x,y) = \lim_{k\to\infty} N_k(x,y) = K(x,y), \qquad x, y \in I.$$

Since

$$\gamma\left(M_0(x,y)\right)+\gamma\left(N_0(x,y)\right)=\gamma(x)+\gamma(y), \qquad x,y\in I,$$

arguing along the same lines as in Lemma 3, we can easily show that

$$\gamma\left(M_k(x,y)\right)+\gamma\left(N_k(x,y)\right)=\gamma(x)+\gamma(y), \qquad k\in\mathbb{N}, x,y\in I.$$

Letting $k \to \infty$ gives

$$2\gamma(K(x,y)) = \gamma(x)+\gamma(y), \qquad x,y\in I. \tag{4}$$

Suppose first that the function $\gamma$ is constant on a nonempty interval $J \subset I$. Since $K$ is a strict mean, the continuity of $K$ and (4) easily imply that $\gamma$ is constant on $I$. By the definition of $\gamma$, there are $a_1,\, b_1 \in \mathbb{R}$ ($a_1 > 0$) such that $\phi(x) = a_1 x + b_1$, $x \in I$, and, consequently,

$$\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) = \frac{x+y}{2}, \qquad x,\, y \in I.$$

Hence, making use of (1), we get

$$\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = \frac{x+y}{2}, \qquad x,\, y \in I,$$

and, by Remark 4,

$$\psi(x) = a_2 x + b_2, \qquad x \in I,$$

for some $a_2$, $b_2 \in \mathbb{R}$, $a_2 \neq 0$.

If $\gamma$ is not constant then there exists an $x_0 \in I$ such that $\phi^{''}(x_0) \neq 0$. Without any loss of generality one can assume that $\phi^{''}(x_0) > 0$. By the continuity of $\phi^{''}$ there exists a maximal subinterval $J \subset I$, open in $I$, such that $x_0 \in J$ and $\phi^{''}(x) > 0$ for all $x \in J$. Hence $\gamma' = \phi^{''}/\phi'$ is positive on $J$ and, therefore, $\gamma$ is strictly increasing on $J$. As $K$ is a mean, we have $K : J^2 \to J$. From (4) we infer that

$$K(x,y) = \gamma^{-1}\left(\frac{\gamma(x)+\gamma(y)}{2}\right), \qquad x, y \in J.$$

On the other hand, in view of Lemma 3, we have

$$K(x,y) = \frac{x+y}{2}, \qquad x, y \in J,$$

and, consequently,

$$\gamma^{-1}\left(\frac{\gamma(x)+\gamma(y)}{2}\right) = \frac{x+y}{2}, \qquad x, y \in J.$$

It follows that $\gamma(x) = ax + c$, $x \in J$, for some $a$, $c \in \mathbb{R}$, $a \neq 0$. Making use of the definition of $\gamma$ we obtain

$$\phi'(x) = e^c e^{ax}, \qquad x \in J.$$

Hence, setting $c_1 := e^c/a$ ($c_1 \neq 0$), we infer that there is $b_1 \in \mathbb{R}$ such that

$$\phi(x) = c_1 e^{ax} + b_1, \qquad x \in J.$$

Note that this formula and the maximality of the subinterval $J$ imply that $J = I$. In fact, in the opposite case one of the endpoints of $J$, say $\alpha$, would be an interior point of $I$ and $\phi^{''}(\alpha) = 0$. Since $\phi^{''}(x) = a^2 c_1 e^{ax}$ for all $x \in J$, by the continuity of $\phi^{''}$, we would get $a^2 c_1 = 0$ which is a contradiction. Thus

$$\phi(x) = c_1 e^{ax} + b_1, \qquad x \in I,$$

and, consequently,

$$\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) = \log\left(\frac{e^{ax}+e^{ay}}{2}\right)^{1/a}, \qquad x, y \in I.$$

Now equation (1) and the condition $\phi^{''} \neq 0$ in $I$ imply that $\psi$ is not an affine function on $I$. Thus $\psi^{''} \not\equiv 0$ on $I$ and therefore there is a maximal subinterval of

$J$ on which $\psi''$ does not vanish. Arguing along the same lines as in the case of function $\phi$, we can prove that there are $b$, $c_2$, $b_2 \in \mathbb{R}$, $b \neq 0 \neq c_2$, such that

$$\psi(x) = c_2 e^{bx} + b_2, \qquad x \in I.$$

Since, by Lemma 1, $\phi'\psi'$ is constant, we infer that $b = -a$. Thus

$$\psi(x) = c_2 e^{-ax} + b_2, \qquad x \in I,$$

and, consequently,

$$\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = \log\left(\frac{e^{-ax}+e^{-ay}}{2}\right)^{-1/a}, \qquad x, y \in I.$$

On the other hand for the functions $\phi$ and $\psi$ given by

$$\phi(x) = c_1 e^{ax} + b_1, \qquad \psi(x) = c_2 e^{-ax} + b_2,$$

where $a$, $c_1$, $c_2 \in \mathbb{R} \setminus \{0\}$, $b_1$, $b_2 \in \mathbb{R}$, are arbitrary, we have

$$\begin{aligned} \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) &+ \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) \\ = \log\left(\frac{e^{ax}+e^{ay}}{2}\right)^{1/a} &+ \log\left(\frac{e^{-ax}+e^{-ay}}{2}\right)^{-1/a} \\ = \frac{1}{a}\cdot\log\frac{e^{ax}+e^{ay}}{e^{-ax}+e^{-ay}} &= \frac{1}{a}\cdot\log e^{ax}e^{ay} = x + y \end{aligned}$$

for all $x$, $y \in \mathbb{R}$. This completes the proof. □

In this context we pose the following open

**Problem 1.** Determine all continuous one-to-one solutions $\phi$, $\psi : I \to \mathbb{R}$ of the functional equation (1).

**Remark 5.** Note that applying the main result of the recent paper [8] one can show that if the bijective functions $\phi$, $\psi$ satisfy equation (1) and, for instance $\phi$ is continuous and $\psi$ is continuous at least at one point, then $\psi$ is continuous.

**Remark 6.** (cf. Aczél and Dhombres [2], p. 246). Let $I \subset \mathbb{R}$ be an interval and $\phi$, $\psi : I \to \mathbb{R}$ arbitrary bijections of $I$ onto some intervals $\phi(I)$ and $\psi(I)$. Then

$$\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right), \qquad x, y \in I,$$

if, and only if, the functions $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are Jensen affine on $\phi(I)$ and $\psi(I)$, respectively. If moreover $\psi \circ \phi^{-1}$ or $\phi \circ \psi^{-1}$ is bounded above or below on a set of a positive measure then there exist $a$, $b \in \mathbb{R}$ such that

$$\psi(x) = a\psi(x) + b, \qquad x \in I.$$

Applying Theorem 1 we obtain the following

**Corollary 1.** *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\phi$, $\psi : I \to \mathbb{R}$ are one-to-one and at least one of them is twice continuously differentiable. Then $\phi$ and $\psi$ satisfy the functional equation*

$$A\left(M_\phi(x,y), M_\psi(x,y)\right) = A(x,y), \qquad x, y \in I,$$

*if, and only if, one of the following two cases occurs*

$1^0$ $\quad$ *$M_\phi = A$ and $M_\psi = A$;*

$2^0$ $\quad$ *there exists $a \in \mathbb{R} \setminus \{0\}$, such that for all $x$, $y \in I$,*

$$M_\phi(x,y) = \log\left(\frac{e^{ax} + e^{ay}}{2}\right)^{1/a}, \qquad M_\psi(x,y) = \log\left(\frac{e^{-ax} + e^{-ay}}{2}\right)^{-1/a}.$$

Inspired by Corollary 1 and Remark 4 we can indicate a more general class of solutions of equation (1) in the case $I = \mathbb{R}$. The following result is easy to verify.

**Proposition 2.**

$1^0$. *If $\phi$, $\psi : \mathbb{R} \to \mathbb{R}$ are Jensen affine and bijective then*

$$\phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = x + y, \qquad x, y \in \mathbb{R}. \tag{5}$$

$2^0$. *For every Jensen affine bijective function $\alpha : \mathbb{R} \to \mathbb{R}$, the functions*

$$\phi = \exp \circ \alpha, \qquad \psi = \exp \circ (-\alpha),$$

*satisfy equation* (5).

As it is an open question if this proposition gives all solutions of equation (5), we pose the following open

**Problem 2.** Determine all bijections $\phi$, $\psi : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation (5).

**Remark 7.** Note that equation (1) is related to the bisymmetric functional equation

$$M\left(N(x,y),K(z,w)\right)=M\left(N(x,z),K(y,w)\right),$$

where $M$, $N$, $K: I^2 \to I$ are means on $I$. Suppose that $M$, $N$, $K$ are quasi-arithmetic means with generators $\phi$, $\psi$ and $\gamma$, respectively. Setting $M=A_\phi$, $N=A_\psi$, $K=A_\gamma$ into the bissymetry equation, and then taking $z=x$ and $w=y$ gives

$$\phi\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)+\phi\left(\gamma^{-1}\left(\frac{\gamma(x)+\gamma(y)}{2}\right)\right)=\phi(x)+\phi(y), \qquad x,y\in I.$$

Put $J:=\phi(I)$ and take arbitrary $u$, $v\in J$. Replacing $x$ and $y$ in this equation respectively by $\phi^{-1}(u)$ and $\phi^{-1}(v)$ gives

$$\begin{aligned}&\phi\left(\psi^{-1}\left(\frac{\psi\left[\phi^{-1}(u)\right]+\psi\left[\phi^{-1}(v)\right]}{2}\right)\right)\\&\quad+\phi\left(\gamma^{-1}\left(\frac{\gamma\left[\phi^{-1}(u)\right]+\gamma\left[\phi^{-1}(v)\right]}{2}\right)\right)=u+v.\end{aligned}$$

Putting $f:=\psi\circ\phi^{-1}$ and $g:=\gamma\circ\phi^{-1}$ we hence get the functional equation

$$f^{-1}\left(\frac{f(u)+f(v)}{2}\right)+g^{-1}\left(\frac{g(u)+g(v)}{2}\right)=u+v, \quad u,\, v\in J,$$

i.e. an equation of the type (1).

## 3. Complementary quasi-arithmetic means with respect to an arbitrary quasi-arithmetic mean

Applying Theorem 1 we can prove the following more general

**Theorem 2.** *Let $I\subset\mathbb{R}$ be an interval. Suppose that $\phi$, $\psi$, $\gamma: I\to\mathbb{R}$ are one-to-one, $\gamma(I)$ is an interval, and at least one of the functions $\phi\circ\gamma^{-1}$, $\psi\circ\gamma^{-1}$ is twice continuously differentiable on $\gamma(I)$. Then $\phi$, $\psi$, $\gamma$ satisfy the functional equation*

$$M_\gamma(M_\phi(x,y),M_\psi(x,y))=M_\gamma(x,y), \qquad x,y\in I, \tag{6}$$

*if, and only if, one of the following two cases occurs:*

$1^0$ *there exist $a_1$, $a_2$, $b_1$, $b_2\in\mathbb{R}$, $a_1\neq 0\neq a_2$, such that*

$$\phi(x)=a_1\gamma(x)+b_1, \qquad \psi(x)=a_2\gamma(x)+b_2, \qquad x\in I;$$

$2^0$ *there exist* $a,\ c_1,\ c_2 \in \mathbb{R}\setminus\{0\},\ b_1,\ b_2 \in \mathbb{R}$, *such that*

$$\phi(x) = c_1 e^{a\gamma(x)} + b_1, \psi(x) = c_2 e^{-a\gamma(x)} + b_2, \qquad x \in I.$$

*Proof.* By the definition of a quasi-arithmetic mean we can write equation (6) in the form

$$\gamma\left(\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)\right) + \gamma\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right) = \gamma(x)+\gamma(y), \qquad x,\ y \in I.$$

By assumption, $J := \gamma(I)$ is an interval. Taking arbitrary $u,\ v \in J$ and setting $x = \gamma^{-1}(u)$, $y = \gamma^{-1}(v)$, gives the following equivalent form of equation (6):

$$\gamma\circ\phi^{-1}\left(\frac{\phi\circ\gamma^{-1}(u)+\phi\circ\gamma^{-1}(v)}{2}\right) + \gamma\circ\psi^{-1}\left(\frac{\psi\circ\gamma^{-1}(u)+\psi\circ\gamma^{-1}(v)}{2}\right) = u+v, \quad u,\ v \in I.$$

Applying Theorem 2 with $I$, $\phi$ and $\psi$ replaced respectively by $J$, $\phi\circ\gamma^{-1}$, $\psi\circ\gamma^{-1}$, we conclude the proof. □

**Remark 8.** Note that a solution $(\gamma, \phi, \psi)$ of equation (6) depends on an arbitrary one-to-one function $\gamma : I \to \mathbb{R}$ such that $\gamma(I)$ is an interval. Theorem 2 also gives very irregular solutions of equation (6). To show this consider the following

**Example 1.** Let $I = \mathbb{R}$ and let $\gamma : \mathbb{R} \to \mathbb{R}$ be an arbitrary discontinuous additive bijection. Then $\gamma(I) = \mathbb{R}$ and, obviously, all the assumptions of Theorem 2 are fulfilled.

As an immediate consequence of Theorem 2 we get

**Corollary 2.** *Let* $I \subset \mathbb{R}$ *be an interval. Suppose that* $\phi,\ \psi,\ \gamma : I \to \mathbb{R}$ *are one-to-one,* $\gamma(I)$ *is an interval, and at least one of the functions* $\phi\circ\gamma^{-1}$, $\psi\circ\gamma^{-1}$ *is twice continuously differentiable on* $\gamma(I)$. *Then* $\phi,\ \psi,\ \gamma$ *satisfy the functional equation*

$$M_\gamma\left(M_\phi(x,y), M_\psi(x,y)\right) = M_\gamma(x,y), \qquad x, y \in I, \tag{7}$$

*if, and only if, one of the following two cases occurs*

$1^0$ $M_\phi = M_\gamma$ *and* $M_\psi = M_\gamma$;

$2^0$ $M_\phi = M_{\exp\circ\gamma}$ *and* $M_\psi = M_{\exp\circ(-\gamma)}$.

**Remark 9.** Note that in Corollary 2. $2^0$, for all $x,\ y \in I$,

$$M_\phi(x,y) = \gamma^{-1}\circ\log\left(\frac{e^{\gamma(x)}+e^{\gamma(y)}}{2}\right)$$

$$M_\psi(x,y) = \gamma^{-1} \circ \left( -\log \left( \frac{e^{-\gamma(x)} + e^{-\gamma(y)}}{2} \right) \right).$$

**Example 2.** Taking in Corollary 2. $2^0$

$$I = (0,\infty), \quad p \in \mathbb{R} \setminus \{0\}, \quad \gamma = p \cdot \log,$$

gives the following solutions of equation (7) :

$$M_\gamma = G, \quad M_\phi(x,y) = \left( \frac{x^p + y^p}{2} \right)^{1/p}, \quad M_\psi(x,y) = \left( \frac{x^{-p} + y^{-p}}{2} \right)^{-1/p},$$

for all $x$, $y > 0$.

**Example 3.** Taking $I = \mathbb{R}$, $\gamma = \exp$, in Corollary 2. $2^0$ gives

$$M_\gamma(x,y) = \log \left( \frac{\exp(x) + \exp(y)}{2} \right),$$

$$M_\phi(x,y) = \log \circ \log \left( \frac{\exp \circ \exp(x) + \exp \circ \exp(y)}{2} \right),$$

$$M_\psi(x,y) = \log \circ (-\log) \left( \frac{\exp \circ (-\exp)(x) + \exp \circ (-\exp)(y)}{2} \right), \quad x,\ y \in \mathbb{R}.$$

## 4. A higher dimensional case

Let $I \subset \mathbb{R}$ be an interval and let $k \in \mathbb{N}$, $k \geq 2$ be fixed. A function $M : I^k \to I$ is said to be a *mean* on $I^k$ if for all $x_1, \ldots, x_k \in I$,

$$\min(x_1, \ldots, x_k) \leq M(x_1, \ldots, x_k) \leq \max(x_1, \ldots, x_k).$$

A mean $M$ is called *strict* if for all $x_1, \ldots, x_k \in I$ such that $x_i \neq x_j$, for some $i$, $j \in \{1, \ldots, k\}$, these inequalities are strict.

Let $M_1, \ldots, M_k : I^k \to I$ be continuous and strict means. One can show (cf. [9]) that there exists a unique $(M_1, \ldots, M_k)$-*invariant mean* $K : I^k \to I$, i.e. such that for all $x_1, \ldots, x_k \in I$,

$$K\left(M_1(x_1, \ldots, x_k), \ldots, M_k(x_1, \ldots, x_k)\right) = K(x_1, \ldots, x_k).$$

A mean $M : I^k \to I$ is called *quasi-arithmetic* if there is a continuous one-to-one function $\phi : I \to \mathbb{R}$ (called a generator of $M$) such that $M = M_\phi$ where

$$M_\phi\left(x_1, \ldots, x_k\right) := \phi^{-1} \left( \frac{\phi(x_1) + \ldots + \phi(x_k)}{k} \right), \qquad x_1, \ldots, x_k \in I.$$

The problem of determining a quasi-arithmetic mean on $I^k$ which is invariant with respect to a system of quasi-arithmetic means $(M_1, \dots, M_k)$ leads to the functional equation

$$M_{\phi_1}(x_1, \dots, x_k) + \dots + M_{\phi_k}(x_1, \dots, x_k) = x_1 + \dots + x_k, \qquad x_1, \dots, x_k \in I,$$

where $\phi_i : I \to \mathbb{R}$, $i = 1, \dots, k$, are unknown functions.

**Lemma 4.** *Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 3$. If $\phi_i : I \to \mathbb{R}$, $i = 1, \dots, k$, are one-to-one and twice differentiable functions satisfying the equation*

$$\sum_{j=1}^{k} M_{\phi_j}(x_1, \dots, x_k) = \sum_{j=1}^{k} x_j, \qquad x_1, \dots, x_k \in I,$$

*then there exist $a_i$, $b_i \in \mathbb{R}$, $a_i \neq 0$, $i = 1, \dots, k$, such that*

$$\phi_i(x) = a_i x + b_i, \qquad i = 1, \dots, k, x \in I.$$

*Proof.* Denote by $Z$ the set of all $x \in I$ such that $\phi'(x) = 0$ for some $i \in \{1, \dots, k\}$. The continuity of $\phi_i'$ and the strict monotonicity of all $\phi_i$ imply that $Z$ is closed in $I$ and $\operatorname{int}(Z) = \emptyset$. Let $J$ be a maximal interval such that $J \subset I \setminus Z$. As $M_{\phi_j}$ are means we have

$$\sum_{j=1}^{k} M_{\phi_j}(x_1, \dots, x_k) = \sum_{j=1}^{k} x_j, \qquad x_1, \dots, x_k \in J.$$

Differentiation of both sides with respect to $x_i$, $i = 1, \dots, k$, gives

$$\sum_{j=1}^{k} \frac{\phi_j'(x_i)}{\phi_j'\left(M_{\phi_j}(x_1, \dots, x_k)\right)} = 0, \qquad x_1, \dots, x_k \in J.$$

Subtracting these equations for $i = 1$ and $i = 2$ by sides we obtain

$$\sum_{j=1}^{k} \frac{\phi_j'(x_2) - \phi_j'(x_1)}{\phi_j'\left(M_{\phi_j}(x_1, \dots, x_k)\right)} = 0$$

for all $x_1, \dots, x_k \in J$. Dividing both sides by $x_2 - x_1$ and letting $x_2$ tend to $x_1$ we get

$$\sum_{j=1}^{k} \frac{\phi_j''(x_1)}{\phi_j'\left(M_{\phi_j}(x_1, x_1, x_3 \dots, x_k)\right)} = 0, \qquad x_1, x_2, \dots, x_k \in J.$$

Differentiation of both sides with respect to $x_3$ gives

$$-\frac{1}{k}\sum_{j=1}^{k}\phi_j^{''}(x_1)\,\frac{\phi_j^{''}\left(M_{\phi_j}(x_1,x_1,x_3\ldots,x_k)\right)}{\left[\phi_j'\left(M_{\phi_j}(x_1,x_1,x_3\ldots,x_k)\right)\right]^2}\,\frac{\phi_j^{'}(x_3)}{\phi_j'\left(M_{\phi_j}(x_1,x_1,x_3\ldots,x_k)\right)}=0$$

for all $x_1, x_3, \ldots, x_k \in J$. Hence, setting $x_1 = x_3 = \ldots = x_k = x$, we get

$$\sum_{j=1}^{k}\left(\frac{\phi_j^{''}(x)}{\phi_j^{'}(x)}\right)^2 = 0, \qquad x \in J,$$

and, consequently, $\phi_j^{''}(x) = 0$ for all $i \in \{1, \ldots, k\}$ and $x \in J$. It follows that there exist $a_i$, $b_i \in \mathbb{R}$, $i = 1, \ldots, k$, such that $\phi_i(x) = a_i x + b_i$, $i = 1, \ldots, k$, $x \in I$. From the strict monotonicity of the functions $\phi_i$, we infer that $a_i \neq 0$, $i = 1, \ldots, k$. Now the differentiability of $\phi_i$, $i = 1, \ldots, k$, implies that $J = I$, and the proof is completed. □

Applying this lemma we get the following

**Theorem 3.** *Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 3$. Suppose that $\phi_1, \ldots, \phi_k, \gamma : I \to \mathbb{R}$, are one-to-one, $\gamma(I)$ is an interval and the functions $\phi_1 \circ \gamma^{-1}, \ldots, \phi_k \circ \gamma^{-1}$ are twice continuously differentiable on $\gamma(I)$. Then $\phi_1, \ldots, \phi_k$ and $\gamma$ satisfy the functional equation*

$$M_\gamma\left(M_{\phi_1}(x_1, \ldots, x_k), \ldots, M_{\phi_k}(x_1, \ldots, x_k)\right) = M_\gamma(x_1, \ldots, x_k), \quad x_1, \ldots, x_k \in I,$$

*if, and only if, there exist $a_i$, $b_i \in \mathbb{R}$, $a_i \neq 0$, $i = 1, \ldots, k$, such that*

$$\phi_i(x) = a_i\gamma(x) + b_i, \qquad (x \in I), \quad i = 1, \ldots, k;$$

*and, consequently,*

$$M_{\phi_i}(x_1, \ldots, x_k) := \gamma^{-1}\left(\frac{\gamma(x_1) + \ldots + \gamma(x_k)}{k}\right), \qquad (x_1, \ldots, x_k \in I),$$

*for $i = 1, \ldots, k$.*

Thus, if $k \geq 3$ the situation is considerably simpler than in the case $k = 2$.

## References

[1] J. Aczél, *Lectures on functional equations and their applications,* Academic Press, New–York–London, 1966.

[2] J. Aczél and J. Dhombres, *Functional equations in several variables,* Cambridge University Press, Cambridge, 1989.

[3] P. Flor and F. Halter-Koch, *Über Folgen, die bezüglich eines Mittels rekurrent sind*, Results Math. *26* (1994), 264–273.

[4] A. Járai, *On regular solutions of functional equations*, Aequationes Math. *30* (1986), 21–54.

[5] P. Kahlig and J. Matkowski, *On the compositions of homogeneous quasi-arithmetic means*, J. Math. Anal. Appl. *216* (1997), 69–85.

[6] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality,* Uniw. Śląski P.W.N., Warszawa–Kraków–Katowice, 1985.

[7] J. Matkowski and M. Wróbel, *A generalized a-Wright convexity and related functional equation*, Ann. Math. Silesianae *10* (1996), 7–12.

[8] J. Matkowski, *Continuous solutions of a functional equation*, Publ. Math. Debrecen *52* (1998), 559–562.

[9] J. Matkowski, *Iterations of mean-type mappings and invariant means*, (to appear).

[10] Zs. Páles, *On two variable functional inequality*, C. R. Math. Rep. Acad. Sci. Canada, *10* (1988), 25–25.

J. Matkowski
Institute of Mathematics
Pedagogical University
Plac Słowiąski 9
PL–65-069 Zielona-Góra

Institute of Mathematics
Silesian University
Bankowa 14
PL–40007 Katowice, Poland

Manuscript received: March 12, 1998 and, in final form, September 12, 1998.