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A CHARACTERIZATION OF L^p-NORM WITH THE AID OF EQUALITY CONDITION IN THE HÖLDER INEQUALITY

Introduction

For a measure space (Ω, Σ, μ) denote by $\boldsymbol{S} = \boldsymbol{S}(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable step functions $x : \Omega \mapsto \mathbb{R}$, and by $\boldsymbol{S}_+ = \boldsymbol{S}_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $x \in \boldsymbol{S}(\Omega, \Sigma, \mu)$. It is easy to see that for an arbitrary bijection $\phi : (0, +\infty) \mapsto (0, +\infty)$ the functional $\boldsymbol{p}_\phi : \boldsymbol{S} \mapsto (0, +\infty)$ given by

$$\boldsymbol{p}_\phi(x) := \begin{cases} \phi^{-1}(\int_{\Omega(x)} \phi \circ |x| d\mu) & \text{if } \mu(\Omega(x)) > 0 \\ 0 & \text{if } \mu(\Omega(x)) = 0, \end{cases} \qquad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),$$

where $\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}$, is well defined (cf. [3], [4]).

Note that for $\phi(t) := \phi(1)t^p,\ t > 0$, where $p \in \mathbb{R} \setminus \{0\}$ is arbitrary and fixed, we have

$$\boldsymbol{p}_\phi(x) = \Big(\int_{\Omega(x)} |x|^p \, d\mu \Big)^{\frac{1}{p}}, \quad x \in \boldsymbol{S}(\Omega, \Sigma, \mu).$$

In [4] the following converse of the Hölder inequality has been proved. Suppose that (Ω, Σ, μ) is a measure space with two sets $A, B \in \Sigma$ such that $0 < \mu(A) < 1 < \mu(B) < \infty$. If $\phi, \psi : (0, \infty) \to (0, \infty)$ are bijections such that

$$(*) \qquad \int_\Omega xy d\mu \leqslant \boldsymbol{p}_\phi(x) \boldsymbol{p}_\psi(y), \quad x, y \in \boldsymbol{S}_+,$$

then ϕ and ψ are conjugate power functions. It is well known that the equality condition in the Hölder inequality occurs if, and only if, the functions x and y are positively proportional.

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In the present paper we show that, under some weak assumptions, if the inequality (*) changes into equality for positively proportional functions, then the bijections ϕ and ψ must be conjugate power functions. Let us mention here that in a recent paper [6] we have shown that the equality in the Minkowski inequality yields a characterization of the L^p-norm.

1. A remark on the definition of p_ϕ and the basic lemma

REMARK 1. Suppose that $\mu(\Omega) > 0$ and take an arbitrary $x \in S_+$ such that $\mu(\Omega(x)) > 0$. Then there exist pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$, of finite and positive measure, and $x_1, \ldots, x_n > 0$, such that

$$x = \sum_{i=1}^{n} x_i \chi_{A_i}.$$

By the definition of $\boldsymbol{p}_\phi$ we get

$$\boldsymbol{p}_\phi(x) = \phi^{-1}\Big(\sum_{i=1}^{n} \phi(x_i)\mu(A_i)\Big).$$

In this paper a crucial role is played by the following

LEMMA 1 ([6], p. 54–55). *Let $\phi : (0,\infty) \to (0,\infty)$ be an arbitrary bijection. Then, for every $a > 0$, the function $\phi \circ (a\phi^{-1})$ is additive if, and only if, the function*

$$(0,\infty) \ni t \mapsto \frac{\phi(t)}{\phi(1)}$$

is multiplicative.

2. Main results

THEOREM 1. *Let (Ω, Σ, μ) be a measure space with A, $B \in \Sigma$ such that $A \cap B = \emptyset$, and $\mu(A)$, $\mu(B)$ are positive and finite. Suppose that ϕ, ψ : $(0,\infty) \to (0,\infty)$ are bijective. If*

$$(1) \qquad y = tx \Rightarrow \int_\Omega xy\, d\mu = \boldsymbol{p}_\phi(x)\boldsymbol{p}_\psi(y), \qquad x \in \boldsymbol{S}_+(\Omega,\Sigma,\mu), \quad t > 0,$$

then $\phi/\phi(1)$ and $\psi/\psi(1)$ are multiplicative. If moreover ϕ and ψ are measurable (or $\log\circ\phi$ and $\log\circ\psi$ are bounded above or below in a neighbourhood of a point), then $\phi/\phi(1)$ and $\psi/\psi(1)$ are conjugate power functions.

Proof. Take an arbitrary set $A \in \Sigma$ such that $0 < \mu(A) < \infty$, and put $a := \mu(A)$. Applying (1) with $x = s\chi_A$, $s > 0$, we have

$$as^2t = \phi^{-1}(a\phi(s))\psi^{-1}(a\psi(ts)), \qquad s, t > 0.$$

Replacing t by $s^{-1}t$ we get

$$ast = \phi^{-1}(a\phi(s))\psi^{-1}(a\psi(t)), \quad s,t > 0,\ a \in M(\Sigma),$$

where $M(\Sigma) := \mu(\Sigma) \setminus \{0, \infty\}$. The bijectivity of ϕ and ψ implies that

$$(2) \qquad a\phi^{-1}(s)\psi^{-1}(t) = \phi^{-1}(as)\psi^{-1}(at), \quad s,t > 0,\ \ a \in M(\Sigma)$$

which can be written in the form

$$\frac{a\phi^{-1}(s)}{\phi^{-1}(as)} = \frac{\psi^{-1}(at)}{\psi^{-1}(t)}, \quad s,\ t > 0; \quad a \in M(\Sigma).$$

It follows that for every $a \in M(\Sigma)$ there exists a $c(a) > 0$ such that

$$a[c(a)]^{-1}\phi^{-1}(s) = \phi^{-1}(as), \quad \psi^{-1}(at) = c(a)\psi^{-1}(t), \quad s,t > 0, \quad a \in M(\mu),$$

or, equivalently,

$$(3) \qquad a\phi(s) = \phi(\frac{a}{c(a)}s), \quad a\psi(t) = \psi(c(a)t), \quad s,t > 0,\ \ a \in M(\mu).$$

Take $A,\ B \in \Sigma$, $A \cap B = \emptyset$, such that $a = \mu(A)$, $b = \mu(B) \in M(\mu)$. Putting $x = u\chi_A + v\chi_B$, $u,\ v > 0$ in (1) gives

$$atu^2 + btv^2 = \phi^{-1}[a\phi(u) + b\phi(v)]\psi^{-1}[a\psi(tu) + b\psi(tv)], \quad u,v,t > 0.$$

By (3) we can write this equation in the form

$$atu^2 + btv^2 = \phi^{-1}\left[\phi\left(\frac{a}{c(a)}u\right) + \phi\left(\frac{b}{c(b)}v\right)\right]\psi^{-1}[\psi(c(a)tu) + \psi(c(b)tv)],$$

for all $u,\ v,\ t > 0$. Replacing u and v respectively by $[c(a)]^{-1}u$ and $[c(b)]^{-1}v$, we obtain, for all $u,\ v,\ t > 0$,

$$(4) \quad \frac{atu^2}{c(a)^2} + \frac{btv^2}{c(b)^2} = \phi^{-1}\left[\phi\left(\frac{a}{c(a)^2}u\right) + \phi\left(\frac{b}{c(b)^2}v\right)\right]\psi^{-1}\left[\psi(tu) + \psi(tv)\right].$$

Taking $t = 1$ we hence get

$$\psi^{-1}\left[\psi(u) + \psi(v)\right] = \frac{\dfrac{au^2}{c(a)^2} + \dfrac{bv^2}{c(b)^2}}{\phi^{-1}\left[\phi\left(\dfrac{a}{c(a)^2}u\right) + \phi\left(\dfrac{b}{c(b)^2}v\right)\right]}, \quad u,v > 0,$$

and, consequently,

$$\psi^{-1}[\psi(tu) + \psi(tv)] = \frac{\dfrac{at^2u^2}{c(a)^2} + \dfrac{bt^2v^2}{c(b)^2}}{\phi^{-1}\left[\phi\left(\dfrac{at}{c(a)^2}u\right) + \phi\left(\dfrac{bt}{c(b)^2}v\right)\right]}, \quad u,v,t > 0.$$

Replacing $\psi^{-1}[\psi(tu)+\psi(tv)]$ in (4) by the right-hand side of the last equation, gives

$$\phi^{-1}\left[\phi\left(\frac{at}{c(a)^2}u\right)+\phi\left(\frac{bt}{c(b)^2}v\right)\right]=t\phi^{-1}\left[\phi\left(\frac{a}{c(a)^2}u\right)+\phi\left(\frac{b}{c(b)^2}v\right)\right],$$

and, after obvious simplification,

$$\phi^{-1}[\phi(tu)+\phi(tv)]=t\phi^{-1}[\phi(u)+\phi(v)], \qquad u,v,t>0.$$

This equation can be written in the equivalent form

$$\phi[t\phi^{-1}(u)]+\phi[t\phi^{-1}(v)]=\phi[t\phi^{-1}(u+v)], \qquad u,v,t>0,$$

which proves that the function $\phi\circ(t\phi^{-1})$ is additive for every $t>0$. By Lemma 1, the function

$$(0,\infty)\ni t\mapsto \phi(t)/\phi(1)$$

is multiplicative. As the roles of the functions ϕ and ψ are symmetric, the function

$$(0,\infty)\ni t\mapsto \psi(t)/\psi(1)$$

is multiplicative too.

The hypothesis on A and B implies that $M(\Sigma)\neq\{1\}$; take $A\cup B$ if necessary. The regularity assumptions on ϕ and ψ imply (cf. for instance [1, p. 41, Theorem 3], also [2, p. 310, Theorem 3]) that there are $p,\, q\in\mathbb{R}$, $p\neq 0\neq q$, such that $\phi(t)=\phi(1)t^p$, $\psi(t)=\psi(1)t^q$, $t>0$. Setting these functions into (2) with $a\neq 1$ we conclude that $p^{-1}+q^{-1}=1$. This completes the proof. ■

In the next result we need the following (cf. [4], p.174)

Definition. The bijective functions $\phi,\, \psi:(0,\infty)\mapsto(0,\infty)$ are said to be *multiplicatively conjugate* iff there are constants $c_1,\, c_2>0$ such that

$$\phi^{-1}(c_1t)\psi^{-1}(c_2t)=t, \qquad t>0.$$

Theorem 2. *Let (Ω,Σ,μ) be a measure space such that*

$$\operatorname{int}(\mu(\Sigma))\neq\emptyset.$$

Suppose that $\phi,\, \psi:(0,\infty)\mapsto(0,\infty)$ are bijective. If

$$y=tx\Leftrightarrow\int_\Omega xy\,d\mu=\boldsymbol{p}_\phi(x)\boldsymbol{p}_\psi\psi(y), \qquad x\in\boldsymbol{S}_+(\Omega,\Sigma,\mu), \quad t>0.$$

then ϕ and ψ are multiplicatively conjugate functions. If moreover one of these functions is measurable or continuous or $\log\circ\phi$ is bounded above (or below) in a neighbourhood of a point, then $\phi/\phi(1)$ and $\psi/\psi(1)$ are conjugate power functions.

Proof. According to the first part of Theorem 1, the functions $\phi/\phi(1)$ and $\psi/\psi(1)$ are multiplicative, and so are their inverses

(5) $$(0,\infty) \ni t \mapsto \phi^{-1}(\phi(1)t), \quad (0,\infty) \ni t \mapsto \psi^{-1}(\psi(1)t).$$

Relation (2) in the proof of Theorem 1 gives

$$a\phi^{-1}(s)\psi^{-1}(t) = \phi^{-1}(as)\psi^{-1}(at), \quad s,t > 0, \ a \in M(\Sigma).$$

Hence, by the multiplicativity of the functions (5) we get

$$\phi^{-1}(\phi(1)a)\psi^{-1}(\psi(1)a) = a, \quad a \in M(\Sigma).$$

Since the function $t \mapsto \phi^{-1}(\phi(1)t)\psi^{-1}(\psi(1)t)$ and the identity function are multiplicative and they coincide on the nonempty interior of the set $\mu(\Sigma)$, they must coincide everywhere, i.e.

$$\phi^{-1}(\phi(1)t)\psi^{-1}(\psi(1)t) = t, \quad t > 0.$$

Thus, according to the definition, the functions ϕ and ψ are multiplicatively conjugate.

The remaining part of the proof is obvious. ∎

REMARK 2. Note that the assumption $\operatorname{int}(\mu(\Sigma)) \neq \emptyset$ in Theorem 2 can be replaced by the following condition: there is a family of measure spaces $((\Omega_i, \Sigma_i, \mu_i))_{i\in I}$ such that

$$\operatorname{int}\Big(\bigcup_{i\in I} \mu_i(\Sigma_i)\Big) \neq \emptyset,$$

and for every $i \in I$

$$y = tx \mapsto \int_\Omega xy\,d\mu = \boldsymbol{p}_\phi(x)\boldsymbol{p}_\psi(y), \quad x \in \boldsymbol{S}_+(\Omega_i, \Sigma_i, \mu_i), \quad t > 0.$$

References

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York–London, 1966.

[2] M. Kuczma, *An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality*, Prace Nauk. Uniw. Śl. 489, Polish Scientific Publishers, Warszawa – Kraków – Katowice, 1985.

[3] J. Matkowski, *On a-Wright convexity and the converse of Minkowski's inequality*, Aequationes Math. 43 (1992), 106–112.

[4] J. Matkowski, *The converse of the Hölder inequality and its generalizations*, Studia Math. 109 (2) (1994), 171–182.

[5] J. Matkowski and J. Rätz, *On the equality condition in Hölder's and Minkowski's inequalities*, Aequationes Math. 46 (1993), 220–228.
[6] J. Matkowski and J. Rätz, *Equality in Minkowski inequality and a characterization of the norm*, Demonstratio Math. 30 (1) (1997), 53–62.

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