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Convex set-valued functions on $(0,\infty)$ and their conjugate

Abstract. Let (Ω,Σ,μ) be a σ-finite space and Y be a Banach space. It is shown that if $F:(0,\infty)\to cl(Y)$ is a convex continuous set-valued function, then

$$\int_\Omega y\left(F\circ\frac{x}{y}\right)d\mu\subset\int_\Omega y\,d\mu\,F\left(\frac{\int_\Omega x\,d\mu}{\int_\Omega y\,d\mu}\right)$$

for all positive μ-integrable functions $x,y:\Omega\to\mathbb{R}$. Moreover, F is convex if and only if its conjugate F^*, $F^*(x)=xF(x^{-1})$, is convex.

It is known that convex functions defined on $(0,\infty)$ are characterized by the inequality

$$(y_1+y_2)f\left(\frac{x_1+x_2}{y_1+y_2}\right)\le y_1f\left(\frac{x_1}{y_1}\right)+y_2f\left(\frac{x_2}{y_2}\right),\tag{1}$$

where $x_1,x_2,y_1,y_2\in(0,\infty)$ (cf. [2], [3], [8]). J. Matkowski ([2], [3]) noticed that this inequality is a simultaneous generalization of the discrete Hölder's and Minkowski's inequalities, and in [4] he obtained an integral version of this inequality which generalizes both the Hölder's and Minkowski's inequalities (cf. also J. Matkowski and J. Rätz [6] where the case of the equality was considered). It was also observed in [3] (cf. also [2]) that through the inequality (1), the function f is strictly related to the function $f^*:(0,\infty)\to\mathbb{R}$, $f^*(x)=xf(x^{-1})$, which is termed the conjugate of f. In this note we present an integral counterpart of (1) for convex set-valued functions, and we show that some basic properties of the conjugate functions remain true for set-valued functions.

Let $\mathbf{Y}$ be a real vector space and $n(\mathbf{Y})$ be the family of all nonempty subsets of $\mathbf{Y}$. Recall that a set-valued function $F:(0,\infty)\to n(\mathbf{Y})$ is convex if its graph is convex or, equivalently, if for all $x,y\in(0,\infty)$ and $t\in(0,1)$,

$$tF(x)+(1-t)F(y)\subset F(tx+(1-t)y).$$

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THEOREM 1

A set-valued function $F : (0, \infty) \to n(\mathbf{Y})$ is convex if and only if

$$y_1 f\left(\frac{x_1}{y_1}\right) + y_2 f\left(\frac{x_2}{y_2}\right) \subset (y_1 + y_2) f\left(\frac{x_1 + x_2}{y_1 + y_2}\right) \tag{2}$$

for all $x_1, x_2, y_1, y_2 \in (0, \infty)$.

Proof. If F is convex then for every positive x_1, x_2, y_1, y_2 we have

$$y_1 F\left(\frac{x_1}{y_1}\right) + y_2 F\left(\frac{x_2}{y_2}\right) = (y_1 + y_2)\left[\frac{y_1}{y_1 + y_2} F\left(\frac{x_1}{y_1}\right) + \frac{y_2}{y_1 + y_2} F\left(\frac{x_2}{y_2}\right)\right]$$
$$\subset (y_1 + y_2) F\left(\frac{x_1 + x_2}{y_1 + y_2}\right).$$

To prove the converse implication take arbitrary $x, y \in (0, \infty)$, $t \in (0, 1)$, and apply (2) with $y_1 = t$, $y_2 = 1 - t$, $x_1 = tx$, and $x_2 = (1 - t)y$.

Now assume that $(\mathbf{Y}, \|\cdot\|)$ is a Banach space and $cl(\mathbf{Y})$ is the family of all closed nonempty subsets of $\mathbf{Y}$. Given a measure space (Ω, Σ, μ) we denote by $\mathbf{L}^1_+(\Omega, \Sigma, \mu)$ the family of all positive μ-integrable functions $x : \Omega \to \mathbb{R}$. For a set-valued function $G : \Omega \to cl(\mathbf{Y})$ the integral $\int_\Omega G\, d\mu$ is understood in the Aumann sense, i.e. it is the set of integrals of all μ-integrable (in Bochner's sense) selections of G.

THEOREM 2

Let (Ω, Σ, μ) be a σ-finite measure space and $\mathbf{Y}$ be a Banach space. If $F : (0, \infty) \to cl(\mathbf{Y})$ is a continuous convex set-valued function, then

$$\int_\Omega y\left(F \circ \frac{x}{y}\right) d\mu \subset \int_\Omega y\, d\mu\, F\left(\frac{\int_\Omega x\, d\mu}{\int_\Omega y\, d\mu}\right) \tag{3}$$

for all $x, y \in \mathbf{L}^1_+(\Omega, \Sigma, \mu)$.

In the proof of this theorem we will use the following fact which for $\mathbf{Y} = \mathbb{R}$ is proved in [6].

LEMMA

Let $y \in \mathbf{L}^1_+(\Omega, \Sigma, \mu)$, $a = \int_\Omega y\, d\mu$, and $\nu(A) = a^{-1} \int_A y\, d\mu$ for all $A \in \Sigma$. If a function $h : \Omega \to \mathbf{Y}$ is μ-integrable, then $\frac{h}{y}$ is ν-integrable and

$$\int_\Omega \frac{h}{y}\, d\nu = a^{-1} \int_\Omega h\, d\mu. \tag{4}$$

Proof. Clearly, ν is a normalized measure absolutely continuous with respect to μ and its Radon-Nikodym derivative $\frac{d\nu}{d\mu} = a^{-1}y$. It is known that a measurable vector-valued function is integrable iff its norm is integrable (cf. [1, Theorem 2, p. 45]). Therefore, by the μ-integrability of h we get that $\|h\|$ is μ-integrable. This implies that $\frac{\|h\|}{y}$ is ν-integrable and, consequently, $\frac{h}{y}$ is ν-integrable. To prove (4) note that for every linear continuous functional $\phi : \mathbf{Y} \to \mathbb{R}$ we have (cf. [6])

$$\int_\Omega \frac{\phi \circ h}{y}\, d\nu = a^{-1} \int_\Omega (\phi \circ h)\, d\mu.$$

Hence

$$\phi\left(\int_\Omega \frac{h}{y}\, d\nu\right) = \phi\left(a^{-1}\int_\Omega h\, d\mu\right),$$

which implies (4).

Proof of Theorem 2. Take $x, y \in \mathbf{L}^1_+(\Omega, \Sigma, \mu)$, put $a := \int_\Omega y\, d\mu$, and consider the measure ν defined by

$$\nu(A) := a^{-1}\int_A y\, d\mu, \quad A \in \Sigma.$$

Let $h : \Omega \to \mathbf{Y}$ be a μ-integrable selection of $y\left(F \circ \frac{x}{y}\right)$. By the Lemma, $\frac{h}{y}$ is a ν-integrable selection of $F \circ \frac{x}{y}$ and

$$a^{-1}\int_\Omega h\, d\mu = \int_\Omega \frac{h}{y}\, d\nu \in \int_\Omega F \circ \frac{x}{y}\, d\nu. \tag{5}$$

The function $\frac{x}{y}$ is also ν-integrable and $\int_\Omega \frac{x}{y}\, d\nu = a^{-1}\int_\Omega x\, d\mu$. Moreover, $\int_\Omega \frac{x}{y}\, d\nu$ is a positive number. Hence, using the integral Jensen inequality for convex functions (cf. [5]) we get

$$\int_\Omega F \circ \frac{x}{y}\, d\nu \subset F\left(\int_\Omega \frac{x}{y}\, d\nu\right) = F\left(a^{-1}\int_\Omega x\, d\mu\right). \tag{6}$$

By (6) and (5) we obtain

$$\int_\Omega h\, d\mu \in \int_\Omega y\, d\mu\, F\left(\frac{\int_\Omega x\, d\mu}{\int_\Omega y\, d\mu}\right).$$

Since h is an arbitrary μ-integrable selection of $y\left(F \circ \frac{x}{y}\right)$, this finishes the proof.

REMARK 1

If values of F are bounded, we can drop the assumption that F is continuous. In that case the convexity of F implies its continuity because the domain of F is finite dimensional (cf. [7, Theorem 3]).

REMARK 2

It is easy to check that if (Ω, Σ, μ) is a non-trivial finite measure space (i.e. there exists an $A \in \Sigma$ such that $0 < \mu(A) < \mu(\Omega)$) and (3) holds for all positive μ-integrable step functions, then F satisfies (2) and hence it is convex.

REMARK 3

Inclusion (3) can be treated as a set-valued generalization of Hölder's and Minkowski's inequalities (cf. [2], [3], [4]).

Given a set-valued function $F : (0, \infty) \to n(\mathbf{Y})$ we define its conjugate $F^* : (0, \infty) \to n(\mathbf{Y})$ by the formula (cf. [2], [3])

$$F^*(x) = xF\left(\frac{1}{x}\right), \quad x \in (0, \infty).$$

Note that the operation "$*$" is an involution i.e.

$$(F^*)^* = F.$$

It is easy to see that F satisfies (2) if and only if F^* does. Therefore as a consequence of Theorem 1 we get the following

COROLLARY

A set-valued function $F : (0, \infty) \to n(\mathbf{Y})$ is convex if and only if F^* is convex.

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