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Characteristic analysis for a polynomial-like iterative equation

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Abstract In the light of Euler's idea for differential equations, a polynomial-like n-order iterative equation is discussed through analyzing its characteristic polynomial. An unproved result is verified rigorously for the first time. Then some conclusions on how the solutions are ruled by those characteristic roots follow.

Keywords: iterative dynamics, functional equation, characteristic.

THE general form of polynomial-like iterative equations

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in R \tag{0.1}$$

is discussed extensively[1—7], where f^k means the kth iterate of $f: R \rightarrow R$ defined by composition of functions, i.e. $f^0 = \mathrm{id}$, the identity, $f^n = \circ f^{n-1}$. A special case is to find iterative roots[8—11], which is a basic problem in dynamical systems of maps on intervals.

One often pays more attention to the particular form:

$$f^n(x) = a^{n-1} f^{n-1}(x) + a_{n-2} f^{n-2}(x) + \ldots + a_0 x, \quad x \in R. \tag{0.2}$$

In 1974 Nabeya[12] gave detailed results of (0.2) for $n = 2$ by considering its characteristics. One is naturally led to study (0.2) generally following him. However, that proves to be difficult. At the 26th International Symposium on Functional Equations held in Spain in 1988, mathematicians presented open problems and conjectures in serial remarks[13], in remark 35 Matkowski proposed that the solutions of (0.2) for $n = k$ are solutions of (0.2) for $n = m$, $m \geqslant k$, if the characteristic polynomial of the lower-order equation exactly divide that of the higher-order one. This gives an important and interesting relation between solutions of the iterative equation and its characteristic roots, but until now no rigorous proof was given.

In this note the above proposed result is proved rigorously. Furthermore, based on this result some conclusions that show the solutions to be ruled by those characteristic roots are given.

1 Characteristic equations

In the light of Euler's idea for differential equations, we formally consider a linear solution

$$f(x) = rx, \quad x \in R \tag{1.1}$$

of eq. (0.2), where $r \in C$ is indeterminate. Then from (0.2) we have

$$P_n(r) := r^n - a_{n-1} r^{n-1} - \cdots - a_1 r - a_0 = 0. \tag{1.2}$$

We call (1.2) the characteristic equation, $P_n(r)$ the characteristic polynomial, and the roots of (1.2) the characteristic roots. By the relation between roots and coefficients eq. (0.2) is equivalent to

$$f^n(x) - \left(\sum_{i=1}^{n} r_i\right) f^{n-1}(x) + \left(\sum_{i<j} r_i r_j\right) f^{n-2}(x) + \cdots + (-1)^n r_1 r_2 \cdots r_n x = 0 \tag{1.3}$$

for $x \in R$, where $r_1, r_2, \ldots, r_n$ are n complex roots of polynomial $P_n(r)$.

Clearly, arbitrarily given n complexes $r_1, r_2, \cdots, r_n \in C$ determine a unique operator $F_n(r_1, r_2, \cdots, r_n)$, defined by

$$F_n(r_1, r_2, \cdots, r_n) f(x) = f^n(x) - \left(\sum_{i=1}^{n} r_i\right) f^{n-1}(x) + \left(\sum_{i<j} r_i r_j\right) f^{n-2}(x)$$

$$+ \cdots + (-1)^n r_1 r_2 \cdots r_n x, \quad x \in R \tag{1.4}$$

for an arbitrary function $f: R \to R$, which maps such a function into another. For convenience we call it n-operator of (1.3).

Lemma 1.1. For fixed $r_1, r_2, \cdots, r_n$ and $r_{n+1} \in C$, if $F_n(r_1, r_2, \cdots, r_n)f \equiv 0$, then $F_{n+1}(r_1, \cdots, r_n, r_{n+1})f \equiv 0$.

Proof. Since $F_n(r_1, r_2, \cdots, r_n)f \equiv 0$, i.e. f satisfies eq. (1.3), we have

$$f^{n+1}(x) = f^n(f(x)) = \left(\sum_{i=1}^{n} r_i\right) f^n(x) - \left(\sum_{i<j}^{n} r_i r_j\right) f^{n-1}(x) + \cdots$$
$$+ (-1)^{n+1} r_1 r_2 \cdots r_n f(x), \quad x \in R. \tag{1.5}$$

Thus for all $x \in R$ the $(n+1)$-operator satisfies

$$\begin{aligned}
&F_{n+1}(r_1, \cdots, r_n, r_{n+1})f(x) \\
&= f^{n+1}(x) - \left(\sum_{i=1}^{n+1} r_i\right) f^n(x) + \left(\sum_{i<j}^{n+1} r_i r_j\right) f^{n-1}(x) + \cdots + (-1)^{n+1} r_1 r_2 \cdots r_{n+1} x \\
&= \left(\sum_{i=1}^{n} r_i - \sum_{i=1}^{n+1} r_i\right) f^n(x) - \left(\sum_{i<j}^{n} r_i r_j - \sum_{i<j}^{n+1} r_i r_j\right) f^{n-1}(x) \\
&\quad + \cdots + (-1)^{n+1} r_1 r_2 \cdots r^{n+1} x \\
&= -r_{n+1} f^n(x) + r_{n+1}\left(\sum_{i=1}^{n} r_i\right) f^{n-1}(x) - r_{n+1}\left(\sum_{i<j}^{n} r_i r_j\right) f^{n-2}(x) \\
&\quad + \cdots + (-1)^{n+1} r_1 r_2 \cdots r_{n+1} x \\
&= -r_{n+1} F_n(r_1, \cdots, r_n) f(x) = 0.
\end{aligned} \tag{1.6}$$

This completes the proof.

Then we can prove Matkowski's remark[13].

Theorem 1.1. Suppose that

$$Q(r) = r^k - b_{k-1} r^{k-1} - \cdots - b_1 r - b_0,$$
$$P(r) = r^n - a_{n-1} r^{n-1} - \cdots - a_1 r - a_0$$

are polynomials, where $r \in C$ and $k \leqslant n$, and that $Q \mid P$, i.e. P is exactly divided by Q. If a function $f: R \to R$ satisfies the iterative equation

$$f^k(x) = b_{k-1} f^{k-1}(x) + b_{k-2} f^{k-2}(x) + \cdots + b_0 x, \quad x \in R, \tag{1.7}$$

then f satisfies the iterative equation

$$f^n(x) = a_{n-1} f^{n-1}(x) + a_{n-2} f^{n-2}(x) + \cdots + a_0 x, \quad x \in R. \tag{1.8}$$

Proof. Let $r_1, r_2, \cdots, r_n$ be complex roots of P. Since $Q \mid P$ we assume without loss of generality that $r_1, \cdots, r_k$, $k \leqslant n$, are roots of Q. From (1.7), the hypothesis is equivalent to

$$F_k(r_1, r_2, \cdots, r_k) f = 0. \tag{1.9}$$

By Lemma 1.1 f also satisfies

$$F_{k+1}(r_1, \cdots, r_k, r_{k+1}) f = 0. \tag{1.10}$$

Thus by induction we can prove easily that

$$F_n(r_1, r_2, \cdots, r_n) f = 0, \tag{1.11}$$

that is, f satisfies eq. (1.8). This proves the theorem.

Indeed Theorem 1.1 gives an important relation between solutions of the iterative equation (0.2) and its characteristic roots. We must point out the following remarks.

Remark 1.1. Eq. (1.8) of order n has a solution which does not satisfy eq. (1.7) of order k if $Q \mid P$ but $Q \neq P$. In fact, if all roots $r_1, r_2, \cdots, r_n$ of P are real and only $r_1, r_2, \cdots, r_k$, $k < n$, are roots of Q, then $f(x) = r_i x$, $x \in R$, $i = k+1, \cdots, n$, satisfies (1.8) but it is not a solution of (1.7).

Remark 1.2. If r_0 is a complex root of P, then all the solutions of the real 2-order iterative equation

$$f^2(x) = 2\mathrm{Re}\, r_0 f(x) - |r_0|^2 x \tag{1.12}$$

satisfy eq. (1.8). Here $\mathrm{Re}\, r_0$ and $|r_0|$ denote respectively the real part and norm of r_0. In fact, all coefficients of P are real, so P must have another root $\overline{r_0}$, the conjugacy of r_0. Then Theorem 1.1 implies our result.

2 Iteration of solutions

For convenience, let

$$F_{n-1}(r_1, \cdots, \check{r_k}, \cdots, r_n) = F_{n-1}(r_1, \cdots, r_{k-1}, r_{k+1}, \cdots, r_n) \tag{2.1}$$

represent the $(n-1)$-operator of (1.3), determined by $n-1$ characteristic roots $r_1, \cdots, r_{k-1}, r_{k+1}, \cdots, r_n$, where $\check{r_k}$ means that r_k is removed from the list of characteristic roots.

Theorem 2.1. Suppose that the characteristic equation (1.2) has n different real roots $r_1 < r_2 < \cdots < r_n$, and that $f: R \to R$ is a solution of the iterative equation (0.2). Then for any integer $m \geqslant 0$,

$$f^{n+m} = \frac{A_{11}}{\Delta} r_1^{m+1} g_1 + \frac{A_{21}}{\Delta} r_2^{m+1} g_2 + \cdots + \frac{A_{n1}}{\Delta} r_n^{m+1} g_n, \tag{2.2}$$

where $g_i := F_{n-1}(r_1, \cdots, \check{r_i}, \cdots, r_n) f$, $i = 1, 2, \cdots, n$. And Δ and A^{i1}, $i = 1, 2, \ldots, n$, denote respectively the determinant and algebraic complement minors of the matrix

$$A = \begin{bmatrix} 1 & -\sum_{i \neq 1} r_i & \sum_{i<j,\, \neq 1} r_i r_j & \cdots & (-1)^{n-1} r_2 r_3 \cdots r_n \\ 1 & -\sum_{i \neq 2} r_i & \sum_{i<j,\, \neq 2} r_i r_j & \cdots & (-1)^{n-1} r_1 r_3 \cdots r_n \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & -\sum_{i \neq n} r_i & \sum_{i<j,\, \neq 1} r_i r_j & \cdots & (-1)^{n-1} r_1 r_2 \cdots r_{n-1} \end{bmatrix}. \tag{2.3}$$

Here $\sum_{i \neq 1}$ and its like denote simply the summations with respect to the indexes from 1 to n with some shown restriction.

Proof. Consider the equivalent equation (1.3), which can be written as

$$f^n - \Big(\sum_{i \neq n} r_i\Big) f^{n-1} + \Big(\sum_{i<\,*,\,\neq\, *} r_i r_j\Big) f^{n-2} + \cdots + (-1)^{n-1} r_1 r_2 \cdots r_{n-1} f$$

$$= r_n f^{n-1} - r_n \Big(\sum_{i \neq n} r_i\Big) f^{n-2} + \cdots + (-1)^{n-1} r_1 r_2 \cdots r_n f^0. \tag{2.4}$$

It follows by the notation of $g_i(x)$ that

$$g_n \circ f = r_n g_n, \tag{2.5}$$

where $\circ$ means composition of functions. Thus for $m \geqslant 0$,

$$g_n \circ f^{m+1} = r_n^{m+1} g_n, \tag{2.6}$$

that is, for fixed n,

$$f^{n+m} - \Big(\sum_{i \neq n} r_i\Big) f^{n+m-1} + \cdots + (-1)^{n-1} r_1 r_2 \cdots r_{n-1} f^{m+1} = r_n^{m+1} g_n, \tag{2.7}$$

which is a linear equation of f^{n+m}, $f^{n+m-1}, \cdots, f^{m+1}$. Similarly, for fixed k, $k = 1, 2, \cdots, n-1$, we get another analogous linear equation. Then we get a system of n linear equations, expressed by

$$AF = G, \tag{2.8}$$

where A is a matrix defined by (2.3), F and G are transposes respectively of the vector $(f^{n+m}, f^{n+m-1}, \cdots, f^{m+1})$ and of the vector $(r_1^{m+1} g_1, r_2^{m+1} g_2, \cdots, r_n^{m+1} g_n)$.

Applying elementary linear transformations repeatedly on the rows of A, without difficulties, we can calculate

$$\Delta = \det A = \prod_{i<j,\, =1} (r_j - r_i). \tag{2.9}$$

Clearly $\Delta \neq 0$, i.e. A is invertible since $r_1, r_2, \cdots, r_n$ are different real numbers. Therefore, the formula (2.2) is turned out directly from eq. (2.8). This completes the proof.

Corrollary 2.1. $r_1, r_2, \cdots, r_n$ are assumed as in Theorem 2.1. If $f: R \to R$ is a solution of a k-order equation of the form (0.2) and its characteristic polynomial exactly divides $P_n(r)$, defined in (1.2), then f^{n+m} is a summation of certain k terms in (2.6). In particular, $f^{n+m} = \frac{A_{i1}}{\Delta} r_i^{m+1} g_i$, for $m \geqslant 0$ when $f(x) = r_i x$.

Proof. In Theorem 2.1 P_n is assumed to have n different real roots $r_1, r_2, \cdots, r_n$. Without loss of generality, we suppose that the first k roots $r_1, r_2, \cdots, r_k$ are just the k roots of the characteristic polynomial associated with the k-order iterative equation which f satisfies, i.e. $F_k(r_1, r_2, \cdots, r_k)f = 0$. By Theorem 1.1,

$$F_{n-1}(r_1, \cdots, r_k, \cdots, \hat{r_i}, \cdots, r_n)f = 0, \quad i = k+1, \cdots, n, \tag{2.10}$$

i.e. $g_i = 0$, $i = k+1, \cdots, n$. By Theorem 2.1,

$$f^{n+m} = \frac{A_{11}}{\Delta} r_1^{m+1} g_1 + \frac{A_{21}}{\Delta} r_2^{m+1} g_2 + \cdots + \frac{A_{k1}}{\Delta} r_k^{m+1} g_k, \quad \forall\, m \geqslant 0. \tag{2.11}$$

Especially when $f(x) = r_i x$, (2.11) implies the result.

In order to discuss the dual equation, the following lemma is useful.

Lemma 2.1. Suppose that $f: R \to R$ is a solution of (0.2). Then (i) f is one-to-one if $a_0 \neq 0$; (ii) f is strictly monotone and onto if $a_0 \neq 0$ and f is continuous.

Proof. If $f(y_1) = f(y_2) = z$, then $f^i(y_1) = f^i(y_2) = f^{i-1}(z)$, $i = 1, 2, \cdots$. Thus

$$f^{n-1}(z) = f^n(y_k) = a_{n-1}f^{n-1}(y_k) + \cdots + a_1 f(y_k) + a_0 y_k, \text{ for } k = 1, 2. \tag{2.12}$$

It follows that $a_0 y_1 = a_0 y_2$, i.e. $y_1 = y_2$ if $a_0 \neq 0$. Hence f is one-to-one.

In addition the function f, one-to-one when $a_0 \neq 0$, must be monotone if f is continuous. To prove (2) it suffices to show that f is onto. Rewrite (0.2) as

$$f^n(x) - a_{n-1}f^{n-1}(x) - \cdots - a_1 f(x) = a_0 x. \tag{2.13}$$

If the interval $I := f(R) \neq R$, without loss of generality we suppose that $\lim_{x\to\infty} f(x) = b$. By continuity of f^k on the whole R, $k = 1, 2, \cdots$, the left-hand side of (2.13) is bounded, but the right-hand side of (2.13) must be unbounded when $x \to \infty$. This contradiction proves that f is onto.

When $a_0 \neq 0$, the relation between roots and coefficients implies that no characteristic root of (0.2) is zero. By Lemma 2.1, (1.3) is equivalent to

$$f^{-n} - \left(\sum_{i=1}^{n} s_i\right) f^{-(n-1)} + \left(\sum_{i<j}^{n} s_i s_j\right) f^{-(n-2)} + \cdots + (-1)^n s_1 s_2 \cdots s_n f^0 = 0, \tag{2.14}$$

where f^{-k} denotes the kth iterate of f^{-1} and $s_i = r_i^{-1}$, $i = 1, 2, \cdots, n$. (2.14) is called the dual equation of (1.3). As for Theorem 2.1 with (2.14) in place of (1.5) we can also prove the following theorem.

Theorem 2.2. Suppose that $a_0 \neq 0$ in (0.2) and that the hypotheses in Theorem 2.1 hold. Then for any integer $m \geqslant 0$,

$$f^{-(n+m)} = \frac{\widetilde{A}_{11}}{\widetilde{\Delta}} s_1^{m+1} \widetilde{g}_1 + \frac{\widetilde{A}_{21}}{\widetilde{\Delta}} s_2^{m+1} \widetilde{g}_2 + \cdots + \frac{\widetilde{A}_{n1}}{\widetilde{\Delta}} s_n^{m+1} \widetilde{g}_n, \tag{2.15}$$

where $\widetilde{g}_i$, $\widetilde{\Delta}$ and $\widetilde{A}_{i1}$ are just modified from g_i, Δ and A_{i1}, $i = 1, 2, \ldots, n$, defined in Theorem 2.1, where r_j is replaced by s_j, $j = 1, 2, \cdots, n$, and f is replaced by f^{-1}.

3 Properties of solutions

In the sequel we always assume that (i) $a_0 \neq 0$; (ii) (0.2) has n different real characteristic roots $r_1 < r_2 < \cdots < r_n$. We shall apply the above theorems to give the properties of solutions. Let f be a real continuous solution of (0.2).

Corollary 3.1. Under the above assumptions, 1° if $-1 < r_1 < r_n < 1$, then f^k approaches 0 as $k \to +\infty$; 2° if $r_1 > 1$ or $r_n < -1$, then f^k approaches 0 as $k \to -\infty$; 3° in both cases 0 is the unique fixed point of f.

Proof. The limit of (2.16) as $m \to +\infty$ gives result 1°. The result 2° is similarly deduced from (2.15). To prove 3°, by reduction to absurdity we assume $f(x_0) = x_0$ for some $x_0 \neq 0$. It follows from (1.3) that

$$1 - \left(\sum_{i=1}^{n} r_i\right) + \left(\sum_{i<j}^{n} r_i r_j\right) + \cdots + (-1)^n r_1 r_2 \cdots r_n = 0, \tag{3.1}$$

which means $\prod_{i=1}^{n}(1 - r_i) = 0$, i.e. at least one of r_i, $i = 1, \cdots, n$, equals 1. This conflicts the hypotheses in 1° and 2°, so 0 is the unique possible fixed point of f.

Furthermore, by Lemma 2.1, f is strictly monotone and onto. If f is decreasing, f must have a fixed point, which should be 0. Otherwise, if f is increasing but $f(0) \neq 0$, without loss of generality, we only discuss the case that $f(0) > 0$ and the other case can be discussed similarly. Since $f(0) > 0$, i.e. $f^{-1}(0) < 0$, we have

$$f(0) < f^2(0) < \cdots < f^k(0) \nrightarrow 0, \quad \text{as} \quad k \to +\infty, \tag{3.2}$$

$$f^{-1}(0) > f^{-2}(0) > \cdots > f^{-k}(0) \nrightarrow 0, \quad \text{as} \quad k \to +\infty, \tag{3.3}$$

which contradict results 1° and 2° respectively, so $f(0) = 0$.

Corollary 3.2. Suppose that $f: R \to R$ is a strictly increasing continuous solution of eq. (0.2). The following results are true.

1° If $-1 < r_1 < \cdots < r_{n-1} < 1 < r_n$ or $r_1 < -1 < r_2 < \cdots < r_n < 1$, and if $f(x) < x$, $\forall x > 0$ and $f(x) > x$, $\forall x < 0$, then f satisfies

$$F_{n-1}(r_1, \cdots, r_{n-1})f = 0 \quad \text{or} \quad F_{n-1}(r_2, \cdots, r_n)f = 0. \tag{3.4}$$

2° If $r_1 < \cdots < r_{n-1} < -1 < r_n < 1$ or $-1 < r_1 < 1 < r_2 < \cdots < r_n$, and if $f(x) > x$, $\forall x > 0$ and $f(x) < x$, $\forall x < 0$, then f satisfies

$$F_{n-1}(r_1^{-1}, \cdots, r_{n-1}^{-1})f^{-1} = 0 \quad \text{or} \quad F_{n-1}(r_2^{-1}, \cdots, r_n^{-1})f^{-1} = 0. \tag{3.5}$$

Proof. Using the same arguments as in Corollary 3.1, we see that 0 is the unique fixed point of f in R in both cases. For simplicity, under the hypotheses of 1° we only discuss the case where $x > 0$ and $-1 < r_1 < \cdots < r^{n-1} < 1 < r_n$. Since f is increasing, for $x > 0$, $x > f(x) > f^2(x) > \cdots > f^k(x) \to 0$ as $k \to +\infty$. By Theorem 2.1, g_n vanishes, i.e. $F_{n-1}(r_1, \cdots, r_{n-1})f = 0$ because $|r_i|^k \to 0$, $i = 1, \cdots, n-1$ but $|r_n|^k$ does not. Similarly we can prove 2° from (2.15) in Theorem 2.2 by considering f^{-1} and r_i^{-1}, $i = 1, \cdots, n$, instead of f and r_i.

Up to now, for the general n-order equation (0.2) the properties of solutions are not yet quite clear except for a few results, e.g. in the above corollaries and in ref. [5]. There is still a lot of work to do.

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