ISSN 0019-5324

Indian Journal of Mathematics

Volume 39, Number 3
December 1997

THE ALLAHABAD MATHEMATICAL SOCIETY
10, C.S.P. Singh Marg, Allahabad, U.P., 211001
India

INDIAN JOURNAL
OF MATHEMATICS
Vol. 39, No. 3, 1997, 275-286

ROUND-OFF STABILITY OF FUNCTIONAL ITERATIONS ON PRODUCT SPACES

JANUSZ MATKOWSKI AND SHYAM LAL SINGH

(*Received* 11 July 1997)

The main purpose of this paper is to obtain certain results concerning the stability of Picard type sequence of iterates of a system of operators on a product space. Stability theorems of Ostrowski, Harder-Hicks and Singh *et al.* are obtained as special cases.

1. Introduction

Let (Y, d) be a metric space and $T : Y \to Y$. In computation, a solution of an equation

$$Tx = x,\ x \in Y \qquad (1.1)$$

is usually approximated by an iterating sequence $\{x_m\} \subset Y$. In practice, an approximate sequence $\{y_m\} \subset Y$ is used in place of $\{x_m\}$. However, in general, both the sequences need not converge to the same point even if y_m converges in Y (see, for instance, Urabe [**12**], Ostrowski [**7**] and Harder-Hicks [**3**], p. 703-705). M. Urabe [**12**] appears to be the first to initiate this kind of stability problem on the real line. A. M. Ostrowski [**7**] seems to be the first to investigate sufficient conditions for the stability of iterating sequences concerning a map T on a metric space (see [**3**, p. 694], [**11**, p. 106] or Remark 4). Recently Harder-Hicks [**3**] (cf. Cor. 3) and Rhoades [**9**] have obtained similar results for a wider class of contractive type maps. Singh *et al.* [**11**] have extended Ostrowski's stability theorem to a contraction-system presented in Matkowski [**5**]- [**6**] (see also [**4**, p. 39], [**10**, p. 794] and Corollary 1).

The purpose of this paper is to extend the main result, Theorem 2, of Harder and Hicks [**3**] to a system of equations on a finite product of metric spaces. Our results (Theorems 3-4) generalize the main theorem, Theorem 3, of [**11**] as well (cf. Remark 4).

1990 *AMS Mathematics Subject Classification* : 65D15, 41A25, 47H10, 54H25.

Key words and phrases: stability of iterating sequences, approximative sequence, round-off error, Banach contraction, contraction system, fixed point.

At the end, we observe that Theorem 3 [11] with certain restrictions is equivalent to Ostrowski's stability theorem (Remark 4).

2. Preliminaries

Let (Y, d) be a metric space and $T: Y \to Y$. Pick x_0 in Y, and let

$$x_{m+1} = f(T, x_m), \quad m = 0, 1, 2, \cdots, \qquad (*)$$

denote some functional iteration procedure which yields a sequence $\{x_m\}$ converging to a fixed point of T. Let $\{y_m\}$ be an arbitrary sequence in Y and, set

$$d(y_{m+1}, f(T, y_m)) \leq \varepsilon_m, \quad m = 0, 1, 2, \cdots \qquad (**)$$

If $\lim_{m\to\infty} \varepsilon_m = 0$ implies that $\lim_{m\to\infty} y_m = p$, then the iteration procedure defined in (*) is called T-stable or stable with respect to T [3] (see also [9] and [11]). (Because of round-off errors involved in ''approximative sequence'' $\{y_m\}$, inequality sign in (**) seems more appropriate than an equality sign in (**)).

Consider the following conditions for $T: Y \to Y$:

(2.1) For each pair of points $x, y \in Y$ at least one of the following is true:

$$d(Tx, Ty) \leq \alpha\, d(x, y),$$

$$d(Tx, Ty) \leq \beta\, [d(x, Tx) + d(y, Ty)],$$

$$d(Tx, Ty) \leq \upsilon\, [d(x, Ty) + d(y, Tx)],$$

where α, β, υ, are non-negative constants satisfying $0 \leq \alpha < 1$, $0 \leq \beta, \upsilon < \frac{1}{2}$.

(2.2) There exists a constant q, $0 \leq q < 1$ such that for each $x, y \in Y$,

$$d(Tx, Ty) \leq q \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

In [8], Rhoades has compared various definitions for contractive type maps that generalize the Banach contraction. He has shown that (2.1) and (2.2) (which are named (19) and (19″) respectively in his paper [8]) are equivalent (see also [3], p. 695). Harder-Hicks [3] have obtained stability results for T satisfying (2.1).

In all that follows, we generally follow the following notations and definitions introduced in Matkowski [5]-[6] (cf. also Czerwik [1], Kuczma *et al.* [4, p. 39] and Singh *et al.* [10]-[11]).

For non-negative numbers a_{ik} $(i, k = 1, \cdots, n)$, let

$$c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for } i \neq k \\ 1 - a_{ik} & \text{for } i = k \end{cases} \tag{2.3}$$

$i, k = 1, \cdots, n$, and $c_{ik}^{(t)}$ are defined recursively by

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,\, k+1}^{(t)} + c_{i+1,\, 1}^{(t)} c_{1,\, k+1}^{(t)} & \text{for } i \neq k \\ c_{11}^{(t)} c_{i+1,\, k+1}^{(t)} - c_{i+1,\, 1}^{(t)} c_{1,\, k+1}^{(t)} & \text{for } i = k\,, \end{cases} \tag{2.4}$$

$i, k = 1, \cdots, n - t - 1$, $t = 0, 1, \cdots, n - 2$. If $n = 1$, we define $c_{11}^{(0)} = a_{11}$.

In [**6**, p. 9-11], it is shown that the system of inequalities

$$\sum_{k=1}^{n} a_{ik}\, r_k < r_i, \quad i = 1, 2, \cdots, n,$$

has a solution $r_i > 0$, $i = 1, 2, \cdots, n$, if and only if

$$c_{ii}^{(t)} > 0, \quad i = 1, \cdots, n - t, \quad t = 0, 1, \cdots, n - 1; \quad n \geq 2 \tag{2.5}$$

hold. Moreover, there exists a number $h \in (0, 1)$ such that

$$\sum_{k=1}^{n} a_{ik}\, r_k \leq h\, r_i, \quad i = 1, 2, \cdots, n, \tag{2.6}$$

for some positive numbers $r_1, r_2, \cdots, r_n$.

Indeed such an h may be obtained (cf. [**1**]) by

$$h = \max_i \left(r_i^{-1} \sum_{k=1}^{n} a_{ik}\, r_k \right). \tag{2.6a}$$

Let (X_i, d_i), $i = 1, 2, \cdots, n$, be metric spaces,

$$X := X_1 \times X_2 \times \cdots \times X_n.$$

and

$$\overset{m}{x} := (\overset{m}{x_1}, \cdots, \overset{m}{x_n}),\ \overset{m}{x_i} \in X_i,\ i = 1, \cdots, n;\ \ m = 0, 1, 2, \cdots.$$

Also

$$x := (x_1, \cdots, x_n),\ x_i \in X_i,\ i = 1, \cdots, n.$$

Thus $y \in X$ means $y = (y_1, \cdots, y_n)$.

The following fixed point theorems are special cases of Theorem 2 of Singh-Gairola [**10**] (see also Remark 1 [**2**]).

THEOREM 1 *Let* (X_i, d_i), $i = 1, \cdots, n$, *be complete metric spaces and* $T_i : X \to X_i$, $i = 1, \cdots, n$, *be such that*

$$d_i\,(T_i\,x, T_i\,y) \le \tag{2.7}$$

$$\max\left\{\sum_{k=1}^{n} a_{ik}\, d_k\,(x_k, y_k),\ b \max\left\{d_i\,(x_i, T_i\,x),\ d_i\,(y_i, T_i\,y),\ \frac{d_i\,(x_i, T_i\,y) + d_i\,(y_i, T_i\,x)}{2}\right\}\right\}$$

for all $x, y \in X$, *where* a_{ik} *and* b *are non-negative numbers,* $b < 1$, *and* (2.3)- (2.5) *hold. Then the system of equations*

$$x_i = T_i\,x,\ \ i = 1, \cdots, n, \tag{2.8}$$

has exactly one solution $p = (p_1, \cdots, p_n)$ *such that* $p_i \in X_i, i = 1, \cdots, n$. *For an arbitrarily fixed* $\overset{0}{x} \in X$, *the sequence of successive approximations*

$$\overset{m+1}{x_i} = T_i\,\overset{m}{x},\ \ i = 1, \cdots, n,\ m = 0, 1, 2, \cdots, \tag{2.9}$$

converges and

$$p_i = \lim_{m \to \infty} \overset{m}{x_i},\ i = 1, \cdots, n. \tag{2.10}$$

THEOREM 2 *Theorem* 1 *with* (2.7) *replaced by the following:*

$$d_i\,(T_i\,x, T_i\,y) \le \tag{2.7L}$$

$$\max\left\{\sum_{k=1}^{n} a_{ik}\, d_k\,(x_k, y_k),\ b \max\left\{\frac{d_i\,(x_i, T_i\,x) + d_i\,(y_i, T_i\,y)}{2},\ \frac{d_i\,(x_i, T_i\,y) + d_i\,(y_i, T_i\,x)}{2}\right\}\right\}.$$

REMARK 1 It is evident that (2.7L) is less general than (2.7), i.e., $T := (T_1, \cdots, T_n)$ satisfying (2.7L) also satisfies (2.7). Notice that (2.7) and (2.7L) are the same if $b = 0$. If $p := (p_1, \cdots, p_n) \in X$ is a solution of (2.8), then p is called a fixed point of T.

REMARK 2 The condition (2.2) is (2.7L) with $(Y, d) = (X_i, d_i)$, $T = T_i$, $i = 1, \cdots, n$, and $n = 1$ such that $q = \max\{a_{11}, b\}$.

COROLLARY 1 (*Matkowski* [5]). *Theorem* 1 *with* $b = 0$.

The next result gives simple conditions under which Corollary 1 can be simply reduced to the classical Banach contraction principle.

COROLLARY 2 *If, in Corollary 1, the matrix* $[a_{ik}]$ *is symmetric then the pair* (X, d) *with* $X := X_1 \times \cdots \times X_n$, *and* $d : X \times X \to R$ *defined by*

$$d(x, y) = \sum_{i=1}^{n} r_i\, d_i\,(x_i, y_i);\;\; x = (x_1, \cdots, x_n),\quad y = (y_1, \cdots, y_n) \in X,$$

is a complete metric space, and $T = (T_1, \cdots, T_n) : X \to X$ *is h-contraction of this metric space, i.e.*

$$d(Tx, Ty) \le h\, d\,(x, y);\quad x, y \in X,$$

where $h \in (0, 1)$ *is given by* (2.6a).

PROOF. Define a metric d on X by

$$d(x, y) = \sum_{i=1}^{n} r_i\, d_i\,(x_i, y_i),\;\; x, y \in X,$$

where r_i, $i = 1, \cdots, n$, are such that (2.6) hold. Then (X, d) is a complete metric space. Let $T : X \to X$ be such that $Tx = (T_1 x, \cdots, T_n x)$. Now it is enough to show that T is a Banach contraction on X. Since T_i, $i = 1, \cdots, n$, satisfy (2.7) with $b = 0$ (cf. Cor. 1), for any $x, y \in X$,

$$d\,(Tx, Ty) = \sum_{i=1}^{n} r_i\, d_i\,(T_i\, x, T_i\, y) \le \sum_{i=1}^{n} r_i \sum_{k=1}^{n} a_{ik}\, d_k\,(x_k, y_k) =$$

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ik}\, r_i \right) d_k\,(x_k, y_k).$$

By the symmetry restriction on a_{ik}'s and (2.6),

$$\sum_{i=1}^{n} a_{ik}\, r_i = \sum_{i=1}^{n} a_{ki}\, r_i \le h\, r_k.$$

Hence

$$d(Tx, Ty) \le \sum_{k=1}^{n} h\, r_k\, d_k\,(x_k, y_k) = h\, d\,(x, y)\,.$$

This completes the proof since $h < 1$.

REMARK 3 An earlier effort to formulate the result of Corollary 2 (see, for instance, [4], p. 49) faces rough weather without the symmetry restrictions on a_{ik}'s.

LEMMA. *If c is a real number such that $b^j \to 0$ as $j \to \infty$, then*

$$\lim_{m\to\infty} \Big(\sum_{j=0}^{m} c^{m-j}\, b^j\Big) = 0.$$

3. Stability Results

The following stability theorems show that the functional iteration for $T := (T_1, \cdots, T_n)$ defined by (2.9) is T-stable whenever T fulfills the requirement of Theorem 1 with $0 \le b < \frac{1}{2}$ of Theorem 2 with no additional requirement.

Notice that the maps considered in (2.1), (2.2), (2.7) and (2.7L) need not be continuous.

THEOREM 3 *Let (X_i, d_i) be complete metric spaces and $T_i : X \to X_i$, $i = 1, \cdots, n$, such that $T := (T_1, \cdots, T_n)$ satisfies the conditions of Theorem* 1, *viz.,* (2.7), (2.3)-(2.5) *and $0 \le b < \frac{1}{2}$. Let $p = (p_1, \cdots, p_n)$ be the fixed point of T. Let x^0 be an arbitrary point in X, and put*

$$x_i^{m+1} = T_i\, x^m, \; m = 0, 1, \cdots, \;\; i = 1, \cdots, n.$$

Let $\{y_i^m\}$ denote an arbitrary sequence in X_i, $i = 1, \cdots, n$, and set

$$d_i\,(y_i^{m+1}, T_i\, y^m) \le \varepsilon_i^m, \; m = 0, 1, \cdots, \;\; i = 1, \cdots, n.$$

Then, for $m = 0, 1, \cdots, \quad i = 1, \cdots, n,$

$$d_i\,(p_i, y_i^{m+1}) \leq d_i\,(p_i, x_i^{m+1}) + \beta^{m+1}\, r_i + c \sum_{j=0}^{m} \beta^{m-1}\, d_i\,(x_i^j, x_i^{j+1}) + \sum_{j=0}^{m} \beta^{m-1}\, \varepsilon_i^j, \quad \text{(I)}$$

where $r_i \geq d_i\,(x_i^0, y_i^0)$ *are some positive numbers,* $c = \dfrac{b}{1-b}$, $\beta = \max\{h, c\}$ *and* h *is defined by* (2.6a). *Also,*

$$\lim_{m \to \infty} y_i^m = p_i \ \textit{iff} \ \lim_{m \to \infty} \varepsilon_i^m = 0. \quad \text{(II)}$$

PROOF. To prove (I), first fix $x, y \in X$. Since $T = (T_1, \cdots, T_n)$ satisfies (2.7), one of the following must hold:

$$d_i\,(T_i\,x, T_i\,y) \leq \sum_{k=1}^{n} a_{ik}\, d_k\,(x_k, y_k); \quad (3.1)$$

$$d_i\,(T_i\,x, T_i\,y) \leq b\, d_i\,(x_i, T_i\,x); \quad (3.2)$$

$$d_i\,(T_i\,x, T_i\,y) \leq b\, d_i\,(y_i, T_i\,y) \leq b\,[d_i\,(y_i, x_i) + d_i\,(x_i, T_i\,x) + d_i\,(T_i\,x, T_i\,y)],$$

that is

$$d_i\,(T_i\,x, T_i\,y) \leq \frac{b}{1-b}\,[d_i\,(x_i, y_i) + d_i\,(x_i, T_i\,x)]; \quad (3.3)$$

$$d_i\,(T_i\,x, T_i\,y) \leq \frac{b}{2}\,[d_i\,(x_i, T_i\,y) + d_i\,(y_i, T_i\,x)] \leq$$

$$\frac{b}{2}\,[d_i(x_i, T_i\,x) + d_i\,(T_i\,x, T_i\,y) + d_i\,(x_i, y_i) + d_i\,(x_i, T_i\,x)],$$

that is

$$d_i\,(T_i\,x, T_i\,y) \leq \frac{b}{2-b}\,[d_i\,(d_i\,(x_i, y_i) + 2\, d_i\,(x_i, T_i\,x)]\,. \quad (3.4)$$

Then from (3.1)-(3.4),

$$d_i\,(T_i\,x, T_i\,y) \leq \quad (3.5)$$

$$\max\left\{\sum_{k=1}^{n} a_{ik}\, d_k(x_k, y_k), \max\left\{\frac{b}{1-b}, \frac{b}{2-b}\right\} d_i(x_i, y_i)\right\} + \max\left\{b, \frac{b}{1-b}, \frac{2b}{2-b}\right\} d_i(x_i, T_i x) \leq$$

$$\max\left\{\sum_{k=1}^{n} a_{ik}\, d_k(x_k, y_k), \beta\, d_i(x_i, y_i)\right\} + c\,\beta\, d_i(x_i, T_i x).$$

This is true for $i = 1, \cdots, n$.

For any m,

$$d_i(p_i, y_i^{m+1}) \leq d_i(p_i, x_i^{m+1}) + d_i(x_i^{m+1}, y_i^{m+1}) + d_i(y_i^{m+1}, T_i y^m)$$

$$\leq d_i(p_i, x_i^{m+1}) + d_i(x_i^{m+1}, y_i^{m+1}) + \varepsilon_i^m. \tag{3.6}$$

Now we estimate the middle term on the right hand side of (3.6). From the homogeneity of the system (2.4), we may assume (see also [**11**], p. 108) without any loss of generality that $d_i(x_i^0, y_i^0) \leq r_i$, for some positive numbers r_i, $i = 1, \cdots, n$, satisfying (2.6). Then from (3.5),

$$d_i(x_i^1, y_i^1) \leq d_i(x_i^1, T_i y^0) + d_i(T_i y^0, y_i^1) \leq d_i(T_i x^0, T_i y^0) + \varepsilon_i^0$$

$$\leq \max\left\{\sum_{k=1}^{n} a_{ik}\, d_k(x_k^0, y_k^0), \beta\, d_i(x_i^0, y_i^0)\right\} + c\, d_i(x_i^0, T_i x^0) + \varepsilon_i^0$$

$$\leq \max\left\{\sum_{k=1}^{n} a_{ik}\, r_h, \beta\, r_i\right\} + c\, d_i(x_i^0, x_i^1) + \varepsilon_i^0$$

$$\leq \max\{h\, r_i, \beta\, r_i\} + c\, d_i(x_i^0, x_i^1) + \varepsilon_i^0$$

$$= \beta\, r_i + c\, d_i(x_i^0, x_i^1) + \varepsilon_i^0 = \lambda_i, \text{ say}$$

(This is true for $i = 1, \cdots, n$.)

Similarly,

$$d_i(\overset{2}{x_i}, \overset{2}{y_i}) \le d_i(T_i \overset{1}{x}, T_i \overset{1}{y}) + \overset{1}{\varepsilon_i}$$

$$\le \max\left\{\sum_{k=1}^{n} a_{ik}\, d_k(\overset{1}{x_k}, \overset{1}{y_k}),\ \beta\, d_i(\overset{1}{x_i}, \overset{1}{y_i})\right\} + c\, d_i(\overset{1}{x_i}, T_i \overset{1}{x}) + \overset{1}{\varepsilon_i}$$

$$\le \max\left\{\sum_{k=1}^{n} a_{ik}\, \lambda_k,\ \beta\, \lambda_i\right\} + c\, d_i(\overset{1}{x_i}, \overset{2}{x_i}) + \overset{1}{\varepsilon_i} \le \beta\, \lambda_i + c\, d_i(\overset{1}{x_i}, \overset{2}{x_i}) + \overset{1}{\varepsilon_i}$$

$$= \beta^2 r_i + c\,\beta\, d_i(\overset{0}{x_i}, \overset{1}{x_i}) + c\, d_i(\overset{1}{x_i}, \overset{2}{x_i}) + \beta\, \overset{0}{\varepsilon_i} + \overset{1}{\varepsilon_i}\,.$$

Inductively

$$d_i(\overset{m+1}{x_i}, \overset{m+1}{y_i}) \le \beta^{m+1} r_i + c \sum_{j=0}^{m} \beta^{m-j}\, d_i(\overset{j}{x_i}, \overset{j+1}{x_i}) + \sum_{j=0}^{m-1} \beta^{m-1-j}\, \overset{j}{\varepsilon_i}\,.$$

Its substitution in (3.6) establishes (I).

To prove (II), first assume $\overset{m}{y_i} \to p_i$, as $m \to \infty$, $i = 1, \cdots, n$. Then, for any i, from (3.5),

$$\overset{m}{\varepsilon_i} \le d_i(\overset{m+1}{y_i}, T_i \overset{m}{y}) \le d_i(\overset{m+1}{y_i}, p_i) + d_i(T_i p, T_i \overset{m}{y}) \le$$

$$d_i(\overset{m+1}{y_i}, p_i) + \max\left\{\sum_{k=1}^{n} a_{ik}\, d_k(p_k, \overset{m}{y_k}),\ \beta\, d_i(p_i, y_i)\right\} + c\, d_i(p_i, T_i p) \le$$

$$d_i(\overset{m+1}{y_i}, p_i) + \left[\left(\sum_{k=1}^{n} a_{ik}\right) + \beta\right] \max\{d_1(p_1, \overset{m}{y_1}), \cdots, d_n(p_n, \overset{m}{y_n})\} + 0.$$

Since each $d_i(p_i, \overset{m}{y_i}) \to 0$ as $m \to \infty$, $\lim_{m\to\infty} \overset{m}{\varepsilon_i} = 0$.

Now suppose $\overset{m}{\varepsilon_i} \to 0$ as $m \to \infty$. Since Theorem 1 guarantees the existence of exactly one solution of (2.8) and, by hypothesis, p is a solution of (2.8), the sequence $\overset{m}{x_i}$ converges to p_i, $i = 1, \cdots, n$, (cf. (2.9)-(2.10)). Recall that $0 < \beta < 1$. Thus from (I),

$$\lim_{m \to \infty} d_i(p_i, y_i^{m+1}) \le$$

$$\lim_{m \to \infty} \left(c \sum_{j=0}^{m} \beta^{m-j} d_i(x_i^j, x_i^{j+1}) \right) + \lim_{m \to \infty} \left(\sum_{j=0}^{m} \beta^{m-j} \varepsilon_i^j \right).$$

Since sequences $\{d_i(x_i^j, x_i^{j+1})\}_{j=0}^{\infty}$ and $\{\varepsilon_i^j\}_{j=0}^{\infty}$ are both convergent to zero, an appeal to the lemma of Harder and Hicks establishes (II).

THEOREM 4 *Let (X_i, d_i) be complete metric spaces and $T_i : X \to X_i$, $i = 1, \cdots, n$, such that $T := (T_1, \cdots, T_n)$ satisfies the conditions of Theorem 2, viz.,* (2.7L), (2.3)-(2.5) and $0 \le b < \frac{1}{2}$. *Let x^0 be an arbitrary point in X, and put*

$$x_i^{m+1} = T_i x^m, \; m = 0, 1, \cdots, \; i = 1, \cdots, n.$$

Let $\{y_i^m\}$ denote an arbitrary sequence in X_i, $i = 1, \cdots, n$, and set

$$d_i(y_i^{m+1}, T_i y^m) \le \varepsilon_i^m, \; m = 0, 1, \cdots, \; i = 1, \cdots, n.$$

Then, for $m = 0, 1, \cdots$, and $i = 1, 2, \cdots, n$,

$$d_i(p_i, y_i^{m+1}) \le d_i(p_i, x_i^{m+1}) + \delta^{m+1} r_i + 2b \sum_{j=0}^{m} \delta^{m-j} d_i(x_i^j, x_i^{j+1}) + \sum_{j=0}^{m} \delta^{m-j} \varepsilon_i^j \quad \text{(III)}$$

where $r_i \ge d_i(x_i^0, y_i^0)$ are some positive numbers, $\delta = \max\{h, b\}$ and h is defined by (2.6a).

PROOF. It may be completed using Theorem 2 and following the proof of Theorem 3.

COROLLARY 3 *Let (Y, d) be a complete metric space, and let $T : Y \to Y$. Assume T* satisfies the condition (2.2). *Let p be the fixed point of T. Let x_0 be an arbitrary point in Y, and put $x_{m+1} = T x_m$ for $m = 0, 1, \cdots$ (so that $\lim_{m\to\infty} x_m = p$). Let y_m denote an arbitrary sequence in Y, and let*

$$d(y_{m+1}, T y_m) \le \varepsilon_m, \; m = 0, 1, \cdots.$$

Then

$$d(p, y_{m+1}) \le d(p, x_{m+1}) + q^{m+1}\, d(x_0, y_0) + 2\, q \sum_{j=0}^{m} q^{m-j}\, d(x_j, x_{j+1}) + \sum_{j=0}^{m} q^{m-j} \epsilon_j ,$$

and $\lim_{m\to\infty} y_m = p$ *if and only if* $\lim_{m\to\infty} \varepsilon_m = 0$.

PROOF. Take $(Y, d) = (X_i, d_i)$, $T = T_i = 1, \cdots, n$, and $n = 1$ in Theorem 4 with $r_1 = d(x_0, y_0)$, $a_{11} = h$ and $q = \delta$, (see also Remark 4).

REMARK 4 Theorems 3 and 4 are the same if $b = 0$. The main result of **[11]** is Theorem 3 with $b = 0$. Ostrowski's classical theorem of stability is derived from Theorem 3 if we take $(Y, d) = (X_i, d_i)$, $T = T_i$, $i = 1, \cdots, n$, and $n = 1$ with $a_{11} = h$, $b = 0$ and $d(x_0, y_0) = r_1$. In view of Corollary 2, Theorem 3 with $b = 0$ and a_{ik} symmetrical is equivalent to the Ostrowski's theorem.

REMARK 5 The main result for the stability of Picard sequence of iterates of Harder-Hicks [3, Th. 2] is the above corollary, (recall that (2.1) and (2.2) are equivalent).

Acknowledgement

The second author thanks Technical University, Bielsko-Biala for its hospitality during his visit in autumn 1995. The authors thank Jolanta Okrzesik for her work in prepairng the typescript.

REFERENCES

1. S. CZERWIK, A fixed point for a system of multivalued transformations, *Proc. Amer. Math. Soc.* **55** (1976) 136-139.
2. U.C. GAIROLA, S.L. SINGH, J.H.M WHITFIELD, Fixed point theorems on product of compact metric spaces, *Demonst. Math.* **28** (1995) 541-548.
3. A.M. HARDER AND T.L. HICKS, Stability results for fixed point iteration procedures, *Math. Japon.* **33** (1988) 693-706.
4. M. KUCZMA, B. CHOCZEWSKI AND R. GER, *Iterative functional euqations* (Encyklopedia of Mathematics and its applications, Vol. 32), Cambridge Univ. Press 1990.
5. J. MATKOWSKI, Some inequalities and a generalization of Banach's principle, *Bull. Acad. Polon. Sci., Sér Sci. Math. Astronom. Phys.* **21** (1973) 323-324.
6. J. MATKOWSKI, Integrable solutions of functional equations, *Dissert. Math. Vol.* CXXVII (Rozprawy Mat.), Warszawa, 1975.
7. A.M. OSTROWSKI, The round-off stability of iterations, *Z. Angew. Math. Mech.* **47** (1967) 77-81.

8. B.E. RHOADES, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.* **226** (1977) 257-290.

9. B.E. RHOADES, Fixed point theorems and stability results for fixed point iteration procedures, *Indian J. Pure Appl. Math.* **21** (1990) 1-9.

10. S.L. SINGH AND U.C. GAIROLA, A general fixed point theorem, *Math. Japon.* **36** (1991) 791-801.

11. S.L. SINGH, S.N. MISHRA AND V. CHADHA, Round-off stability of iterations on product spaces, *C.R. Math. Rep. Acad. Sci. Canada* **16** (1994) 105-109.

12. M. URABE, Convergence of numerical iteration in solutions of equations, *J. Sci. Hiroshima Univ., Ser. A*, **19** (3) (1956) 479-489.

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
WILLOWA 2
43-309 BEILSKO-BIALA, POLAND

GURUKULA KANGRI UNIVERSITY
HARDWAR- 249404
INDIA

Allahabad Mathematical Society, founded on the 25th of November, 1958, by Professor B. N. Prasad, an eminent mathematician and analyst, is a non-profit scientific organisation for promoting the cause of advanced study and research in various branches of mathematics including theoretical physics and mathematical statistics. The Society provides an outlet for original researches through its publications and organises lectures, symposia and conferences.

Publications of the Society

Indian Journal of Mathematics (IJM) : Three numbers per volume are published annually; it contains original research papers in different branches of mathematics and mathematical statistics.

Price : US $ 220 or equivalent thereof in Indian rupees (for institutions running research degree programmes in SAARC countries Indian Rs. 550 for Volume 38 (1996) and Volume 39 (1997); Indian Rs. 1000 for Volume 40 (1998)).

Bulletin of the Allahabad Mathematical Society (BULL. ams) : Contains survey articles on current topics, research announcements, book reviews, abstracts of papers presented at meetings of the Society and reports on the activities of the Society.

Price : US $ 50 or equivalent thereof in Indian rupees (for institutions running research degree programmes in SAARC countries Indian Rs. 200) for Volume 8-9 (1993-94) and Volume 10-11 (1995-1996).

Communications in Theoretical Physics (CTP) : A half yearly journal devoted to research papers in theoretical physics.

Price : US $ 120 or equivalent thereof in Indian rupees (for institutions running research degree programmes in SAARC countries Indian Rs. 300) for Volume 5 (1996).

Lecture Note Series : Numbers in this series form texts/monographs on areas of current interest.

Contributors to Indian Journal of Mathematics should submit three copies of the manuscript to the Editor or Secretary for Publications, Allahabad Mathematical Society, 10 C.S.P. Singh Marg, Allahabad, U.P., 211001, India. Submission of a paper implies that the paper is original and is not submitted elsewhere. Authors are requested to leave margins of at least 5 cms. on both sides of the pages. Special instructions for the typesetter, where necessary, should be marked in light pencil on the left margin. The latest issue of the journal should be consulted for style.

A galley proof will be sent to the first author unless otherwise indicated. Authors are requested to retain a copy of the paper as manuscript is not sent with the proof.

25 reprints are supplied gratis to the first author. Additional reprints can be obtained on payment by placing order at the time of proof correction.

Enquiries regarding subscriptions to the publications (current and back volumes) and membership of the Society should be addressed to :

The Treasurer, Allahabad Mathematical Society, 10, C.S.P. Singh Marg, Allahabad, U.P., 211001, India. (e-mail : *pramila @almas.almas.wiprobt.vsnl.ems. net.in*)

All correspondence of general nature related to the Society and its activities should be addressed to :

The Secretary, Allahabad Mathematical Society, 10, C.S.P. Singh Marg, Allahabad, U.P., 211001, India. (e-mail : *pramila@almas.almas.wiprobt.vsnl.ems.net.in*)

Indian Journal of Mathematics

Volume 39 Number 3 December 1997

CONTENTS

Published by : The Committee for Publications, Allahabad Mathematical Society, 10, C.S.P. Singh Marg, Allahabad, U.P., 211001, India.

Laser Setting : Jain Lasers, 33 Dilkusha New Katra, Allahabad-211002, U.P., India.

Printed by : **Mudran Kendra, 609-A Subhash Nagar, Allahabad-211002, U.P.,**
: India. Ph. : 641879, 642783.