Acta Mathematica Sinica, New Series
1997, July, Vol.13, No.3, pp. 421–432

Method of Characteristic for Functional Equations in Polynomial Form

Janusz Matkowski
(*Department of Mathematics, Technical University of Bielsko-Biala 43–309 Bielsko-Biala, Poland*)
Zhang Weinian
(*Department of Mathematics, Sichung Union University, East Area, Chengdu 610041, China*)

Abstract Properties of continuous solutions of a second order polynomial-like iterated functional equation are given by considering its characteristic. A useful method to discuss the general case is described indeed in this procedure.

Keywords Functional equation, Iterative root, Characteristic equation, Recursive construction
1991MR Subject Classification 39B10,39B22
Chinese Library Classification O175.7, O241.3, O174.1

1 Introduction

The polynomial-like iterated functional equation

$$f^{n+1}(x) = a_n f^n(x) + a_{n-1} f^{n-1}(x) + \cdots + a_0 x, \quad a_0 \neq 0, \tag{1.1}$$

is an important form of functional equations [1-9], where $x \in I$, an interval of $\mathbb{R}$, $f : I \to I$ is an unknown function, f^n denotes the n-th iterate of f, and $a_0, a_1, \cdots, a_n$ are real constants. Many functional equations can be reduced to it. For example, the equation

$$f(2x - f(x)/m) = mx, \quad f \in C^0(\mathbb{R}, \mathbb{R}), \tag{1.2}$$

proposed by Biven [10], is equivalent to

$$h^2(x) = 2h(x) - x, \quad x \in \mathbb{R}, \tag{1.3}$$

by setting $g(x) = f(x)/m$ and $h = g^{-1}$. Also it is worth mentioning that Eq. (1.1), related to the linear difference equation

$$x_{k+n+1} = a_n x_{k+n} + \cdots + a_1 x_{k+1} + a_0 x_k, \tag{1.4}$$

Received March 17, 1994, Revised September 27, 1995, Accepted November 6, 1996
Supported by NSF of China, Sichuan Provincial Youth Sci-Tech Foundation and Math. Grant of CAS

is actually a nonlinear problem because the set of its solutions does not span a linear space. In particular, the known Babbage's functional equation

$$f^n(x) = x, \tag{1.5}$$

concerning iterative roots, is a special case of Eq. (1.1).

In this paper Eq. (1.1) for $n = 1$, *i.e.*,

$$f^2(x) = a_1 f(x) + a_0 x, \quad a_0 \neq 0, a_0, a_1 \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.6}$$

is discussed in detail. Properties of its continuous solutions are given through analysing its characteristic equation. In some cases how its general solutions are constructed is shown. Especially, we discuss the case of noncritical characteristic in Sections 3 and 4, the particular case of equal absolute values in 5, the case of critical characteristic in 6 and that of no real characteristic in 7, after setting up its characteristic theory in 2. In this procedure a useful method to discuss the general case of Eq. (1.1) is described naturally.

2 Characteristic Equation

Setting in Eq. (1.6) that

$$f(x) = rx, \quad r \in \mathbb{C}, \neq 0, \quad \text{to be determined}, \tag{2.1}$$

we deduce an equation

$$r^2 = a_1 r + a_0, \quad a_0 \neq 0, \tag{2.2}$$

which is called the **characteristic equation** of Eq. (1.6). Its root r is called a **characteristic root** and the corresponding function $f(x)$ in (2.1) is called a **characteristic solution** of Eq. (1.6). We shall see how the continuous solutions of Eq. (1.6) depend on its characteristic roots.

Let r_1, r_2 be roots of (2.2). Clearly

$$r_1 + r_2 = a_1, \quad r_1 r_2 = -a_0. \tag{2.3}$$

Thus Eq. (1.6) can be rewritten in the form

$$f^2(x) = (r_1 + r_2) f(x) - r_1 r_2 x, \tag{1.6r}$$

and by the following Lemma 1 we have the **dual equation**

$$f^{-2}(x) = (1/r_1 + 1/r_2) f^{-1}(x) - \frac{1}{r_1 r_2} x. \tag{1.6d}$$

Here $1/r_1$ and $1/r_2$ are clearly the characteristic roots of (1.6d).

Lemma 1 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (1.6). Then* a) *f is one to one;* b) *f is strictly monotone and onto, provided f is continuous.*

Proof If $f(y_1) = f(y_2) = z$,

$$f(z) = a_1 f(y_1) + a_0 y_1 = a_1 f(y_2) + a_0 y_2. \tag{2.4}$$

Thus $a_0y_1 = a_0y_2$ and $y_1 = y_2$. The result a) is proved. By the continuity f must be monotone. To prove b) it suffices to show that f is onto. Let $F(x) = a_0^{-1}(f(x) - a_1x)$. Then

$$F \circ f = id(\text{identity}), \tag{2.5}$$

i.e., $F = f^{-1}$ on $I = f(\mathbb{R})$. Naturally $F(I) = \mathbb{R}$. Suppose $I \neq \mathbb{R}$ and b is a finite endpoint of I. Then $F(x) = f^{-1}(x) \to \infty$ as $x \to b$ in I, so F could not be continuous at the point b. This contradicts the continuity of F defined above on whole $\mathbb{R}$.

Lemma 2 (Iteration of Solutions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of Eq.* (1.6), *r_1 and r_2 be two characteristic roots. Then*

i) *when $r_1 \neq r_2$,*

$$f^n(x) = A_2(n)(f(x) - r_1x) - A_1(n)(f(x) - r_2x), \tag{2.6a}$$

for $n = 0, 1, 2, \cdots$, where $A_i(n) = r_i^n/(r_2 - r_1), i = 1, 2$, and

$$f^{-n}(x) = B_2(-n)(f^{-1}(x) - x/r_1) - B_1(-n)(f^{-1}(x) - x/r_2), \tag{2.6b}$$

for $n = 0, 1, 2, \cdots$, where $B_i(-n) = r_i^{-n}/(1/r_2 - 1/r_1)$;

ii) *when $r_1 = r_2 = r$,*

$$f^n(x) = nr^{n-1}f(x) - (n-1)r^n x, \quad n = 0, 1, 2, \cdots, \tag{2.7a}$$

$$f^{-n}(x) = nr^{1-n}f^{-1}(x) - (n-1)r^{-n}x, \quad n = 0, 1, 2, \cdots. \tag{2.7b}$$

Remark 1 For arbitrarily given $x_0, x_1 \in \mathbb{R}$ defines the sequence $\{x_n\}, n = 0, +1, +2, \cdots$, recursively such that

$$x_{n+2} = (r_1 + r_2)x_{n+1} - r_1r_2x_n, \quad n = 0, 1, \cdots; \tag{2.8a}$$

$$x_{-n-2} = (1/r_1 + 1/r_2)x_{-n-1} - \frac{1}{r_1r_2}x_{-n}, \quad n = -1, 0, 1, \cdots. \tag{2.8b}$$

By Lemma 2 we get the general solution of the difference equation (1.4) for $n = 1$, *i.e.*,

$$x_n = A_2(n)(x_1 - r_1x_0) - A_1(n)(x_1 - r_2x_0), \quad n = 0, 1, \cdots, \tag{2.9a}$$

$$x_{-n+1} = B_2(-n)(x_0 - x_1/r_1) - B_1(-n)(x_0 - x_1/r_2), \quad n = 0, 1, \cdots. \tag{2.9b}$$

Proof From (1.6r), the equivalent form, we have

$$f(f(x)) - r_2f(x) = r_1(f(x) - r_2x), \tag{2.10}$$

that is,

$$g(f(x)) = r_1g(x) \tag{2.11}$$

for short with $g(x) = f(x) - r_2x$. By an easy induction we see that $g(f^n(x)) = r_1^n g(x)$, *i.e.*,

$$f^{n+1}(x) - r_2f^n(x) = r_1^n(f(x) - r_2x). \tag{2.12}$$

Similarly we also get

$$f^{n+1}(x) - r_1f^n(x) = r_2^n(f(x) - r_1x). \tag{2.13}$$

Subtracting (2.12) from (2.13) implies (2.6a). Similarly (2.6b) follows from (2.6d). Furthermore, when $r_1 = r_2 = r$, Eq. (1.6r) and (1.6d) have respectively the form $f^2(x) = 2rf(x) - r^2x$ and the form $f^{-2}(x) = 2r^{-1}f^{-1}(x) - r^{-2}x$. Thus (2.7a) and (2.7b) follow by induction.

Lemma 3 (Rate of Variation) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous solution of Eq.* (1.6), *r_1 and r_2 be characteristic roots with $|r_1| < |r_2|$. Then*

i) *if $r_1 > 0$ and $r_2 > 0$,*

$$r_1 \leq (f(x_2) - f(x_1))/(x_2 - x_1) \leq r_2, \quad \forall x_1 \neq x_2; \tag{2.14}$$

ii) *if $r_1 < 0$ and $r_2 > 0$, when f is increasing*

$$r_2 \leq (f(x_2) - f(x_1))/(x_2 - x_1), \quad \forall x_1 \neq x_2, \tag{2.15}$$

and when f is decreasing

$$f(x) = r_1 x + c \quad \text{for some} \quad c \in \mathbb{R}; \tag{2.16}$$

iii) *if $r_1 > 0$ and $r_2 < 0$, when f is increasing*

$$0 \leq (f(x_2) - f(x_1))/(x_2 - x_1) \leq r_1, \quad \forall x_1 \neq x_2, \tag{2.17}$$

and when f is decreasing

$$f(x) = r_2 x + c \quad \text{for some} \quad c \in \mathbb{R}; \tag{2.18}$$

iv) *if $r_1 < 0$ and $r_2 < 0$,*

$$r_2 \leq (f(x_2) - f(x_1))/(x_2 - x_1) \leq r_1, \quad \forall x_1 \neq x_2. \tag{2.19}$$

Proof For case i), from (2.6a) in Lemma 2 we see that

$$u(x) := \lim_{n\to\infty} f^n(x)/r_2^n = (f(x) - r_1 x)/(r_2 - r_1), \quad \forall x \in \mathbb{R}. \tag{2.20a}$$

Since, according to Lemma 1, f is strictly monotone, the iterates f^n are increasing for even n. As the limit of a sequence of increasing functions $u(x)$ has to be nondecreasing. Thus (2.20a) implies the function $x \mapsto f(x) - r_1 x$ is nondecreasing in $\mathbb{R}$ because $r_2 > r_1$, that is, for $x_1 < x_2$ we have $f(x_1) - r_1 x_1 \leq f(x_2) - r_1 x_2$, so

$$r_1 \leq (f(x_2) - f(x_1))/(x_2 - x_1), \quad \forall x_1 \neq x_2. \tag{2.21a}$$

Similarly from (2.6b) in Lemma 2 we see that

$$v(x) := \lim_{n\to\infty} r_1^n f^{-n}(x) = (x/r_2 - f^{-1}(x))/(1/r_2 - 1/r_1), \quad \forall x \in \mathbb{R}, \tag{2.20b}$$

and $v : \mathbb{R} \to \mathbb{R}$ is nondecreasing. Moreover the function $x \mapsto x/r_2 - f^{-1}(x)$ is nonincreasing becuase $1/r_2 - 1/r_1 < 0$. In view of (2.21a) $f(x)$ is strictly increasing. Thus with a substitution of variables we have $f(x_1)/r_2 - x_1 \geq f(x_2)/r_2 - x_2$ for $x_1 < x_2$, *i.e.*,

$$(f(x_2) - f(x_1))/(x_2 - x_1) \leq r_2, \quad \forall x_1 \neq x_2. \tag{2.21b}$$

Hence (2.21a) and (2.21b) complete the proof of i).

For case ii), $1/r_2 - 1/r_1 > 0$ and thus by (2.20b) the function $x \mapsto x/r_2 - f^{-1}(x)$ is nondecreasing. Hence $f(x_1)/r_2 - x_1 \le f(x_2)/r_2 - x_2$ for $x_1 < x_2$ when f is increasing, and (2.15) is proved in the same way as above. When f is decreasing, f^n is also decreasing for each odd n, and thus $u(x)$ in (2.20a) must be nonincreasing since $r_2 > 0$. However, in the last paragraph $u(x)$ is shown to be nondecreasing, so $u(x)$ should be constant. This together with (2.20a) yields (2.16).

The proofs of iii) and iv) are analogous.

Remark 2 By Lemma 3 i) f and f^{-1} are "strongly monotone", *i.e.*, for $x_1, x_2 \in \mathbb{R}$, $(f(x_1) - f(x_2))(x_1 - x_2) \ge r_1|x_1 - x_2|^2$ and $(f^{-1}(x_1) - f^{-1}(x_2))(x_1 - x_2) \ge r_2|x_1 - x_2|^2$.

Lemma 4 *If the solution of $f : \mathbb{R} \to \mathbb{R}$ of Eq. (1.6) has a nonzero fixed point, then one of its characteristic roots equals 1.*

Proof Assume $f(x_0) = x_0, x_0 \ne 0$. It follows from (1.6) that

$$x_0 = a_1 x_0 + a_0 x_0, \quad i.e. \quad a_1 + a_0 = 1. \tag{2.22}$$

From (2.3) we have $r_1 + r_2 - r_1 r_2 = 1$, *i.e.*,

$$(1 - r_1)(1 - r_2) = 0. \tag{2.23}$$

Hence either $r_1 = 1$ or $r_2 = 1$.

3 Noncritical Cases where $r_1 r_2 > 0$

In this section r_1 and r_2 denote two noncritical real characteristic roots of Eq. (1.6), *i.e.*, $|r_1| \ne 1$ and $|r_2| \ne 1$, under the condition of which the characteristic solutions defined in (2.1) have hyperbolic iterative dynamical behaviors. The discussion will proceed separately for cases (1) $1 < r_1 < r_2$, (2) $0 < r_1 < 1 < r_2$, (3) $0 < r_1 < r_2 < 1$, (4) $r_1 < r_2 < -1$, (5) $r_1 < -1 < r_2 < 0$, and (6) $-1 < r_1 < r_2 < 0$.

Theorem 1 *Suppose $1 < r_1 < r_2$. (i) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of Eq. (1.6) then $f(0) = 0$ and f, strictly increasing, satisfies the "two-side" Lipschitzian condition $r_1 \le (f(x) - f(x'))/(x - x') \le r_2$ for $x \ne x'$ in $\mathbb{R}$. (ii) Conversely, Eq. (1.6) has a continuous solution depending on an arbitrary function. More precisely, for every $x_0 > 0, x_1 > 0$ and $f_0 : [x_0, x_1] \to \mathbb{R}$ such that*

$$r_1 x_0 \le x_1 \le r_2 x_0, \tag{3.1}$$

$$f_0(x_0) = x_1, \quad f_0(x_1) = (r_1 + r_2)x_1 - r_1 r_2 x_0, \tag{3.2}$$

$$r_1 \le (f_0(x) - f_0(x'))/(x - x') \le r_2, \quad \forall x, x' \in [x_0, x_1], \tag{3.3}$$

there is a unique continuous function $p : (0, \infty) \to (0, \infty)$ satisfying Eq. (1.6) on $(0, \infty)$ and $p = f_0$ on $[x_0, x_1]$; for two arbitrary initial functions f_{01} and f_{02} like f_0, the function

$$f(x) := \begin{cases} p_1(x), & x > 0, \\ 0, & x = 0, \\ -p_2(-x), & x < 0. \end{cases} \tag{3.4}$$

is a continuous solution of Eq. (1.6) on $\mathbb{R}$, where p_1 and p_2 are functions like p determined as above by f_{01} and f_{02}. (3.4) gives all continuous solutions of Eq. (1.6) in $\mathbb{R}$.

Proof For given $x_0 > 0$ and $x_1 > 0$ in (3.1), the sequences $\{x_n | n = 1, 2, \cdots\}$ and $\{x_{-n} | n = 1, 2, \cdots\}$, defined by (2.8a) and (2.8b), are strictly monotone and $x_n \to \infty, x_{-n} \to 0$ as $n \to \infty$. From (3.2) and (3.3) we can define recursively homeomorphisms $f_n : [x_n, x_{n+1}] \to [x_{n+1}, x_{n+2}], n = 0, 1, \cdots$, such that

$$f_n(x_n) = x_{n+1}, \quad f_n(x_{n+1}) = x_{n+2}, \tag{3.5}$$

and

$$r_1 \leq (f_n(x) - f_n(x'))/(x - x') \leq r_2, \quad \forall x, x' \in [x_n, x_{n+1}]. \tag{3.6}$$

In fact, for a defined f_n in (3.5) and (3.6) we let

$$f_{n+1}(x) = (r_1 + r_2)x - r_1 r_2 f_n^{-1}(x), \quad \forall x \in [x_{n+1}, x_{n+2}]. \tag{3.7}$$

Obviously (3.5) implies $f_{n+1}(x_{n+1}) = x_{n+2}$ and $f_{n+1}(x_{n+2}) = x_{n+3}$. Moreover, by (3.6) we have $1/r_2 \leq (f_n^{-1}(x) - f_n^{-1}(x'))/(x - x') \leq 1/r_1$ for $x, x' \in [x_{n+1}, x_{n+2}]$. It is not difficult to deduce

$$r_1 \leq (f_{n+1}(x) - f_{n+1}(x'))/(x - x') \leq r_2, \quad \forall x, x' \in [x_{n+1}, x_{n+2}]. \tag{3.8}$$

By induction f_n satisfying (3.5) and (3.6) is well defined. Similarly, we can also define recursively homeomorphisms $f_{-n} : [x_{-n+1}, x_{-n+2}] \to [x_{-n}, x_{-n+1}], n = 0, 1, \cdots$, by the properties of the dual equation (1.6d) such that

$$f_{-n}(x_{-n+1}) = x_{-n} \quad f_{-n}(x_{-n+2}) = x_{-n+1}, \tag{3.5d}$$

$$1/r_2 \leq (f_{-n}(x) - f_{-n}(x'))/(x - x') \leq 1/r_1, \quad \forall x, x' \in [x_{-n+1}, x_{-n+2}]. \tag{3.6d}$$

Finally, define

$$p(x) := \begin{cases} f_n(x), & x \in [x_n, x_{n+1}], \quad n = 0, 1, \cdots, \\ f_{-n}^{-1}(x), & x \in [x_{-n}, x_{-n+1}], \quad n = 1, 2, \cdots. \end{cases} \tag{3.9}$$

Since $f_n(x_{n+1}) = f_{n+1}(x_{n+1}), n = 0, 1, \cdots, f_{-1}^{-1}(x_0) = f_0(x_0)$, and $f_{-n}^{-1}(x_{-n+1}) = f_{-n+1}^{-1}(x_{-n+1})$, $n = 2, 3, \cdots, p : (0, \infty) \to (0, \infty)$ is continuous. By the recursive construction of f_n in (3.7) and f_{-n} in the corresponding dual formulas $p(x)$ satisfies Eq. (1.6). In particular, $p(x)$ can be extended continuously at the end-point 0 such that $p(0) = 0$ since by (3.5d) $p(x_{-n}) = x_{-n+1}$ and $x_{-n} \to 0$ as $n \to \infty$. Furthermore, by Lemma 3 i), the above construction enables us to obtain all continuous solutions of Eq. (1.6) on $(0, \infty)$. Also, we observe that $q : (-\infty, 0) \to (-\infty, 0)$ is a continuous solution of Eq. (1.6) iff the function $p(x) := -q(-x)$ is a continuous solution on $(0, \infty)$. Thus by Lemma 3 i) $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of Eq. (1.6) iff $p := f|(0, \infty)$ and $q := f|(-\infty, 0)$ satisfy Eq. (1.6) on $(0, \infty)$ and $(-\infty, 0)$ respectively and $f(0) = 0$. This completes the proof.

Remark 3 Taking $x_1 = r_1 x_0$ (resp. $x_1 = r_2 x_0$) in Theorem 1 we get, as the only possible solution, $f(x) = r_1 x$ (resp. $f(x) = r_2 x$) for $x \in (0, \infty)$. In fact, in this case there is only one initial function f_0 satisfying (3.1–3.3), namely $f_0(x) = r_1 x$ (resp. $f_0(x) = r_2 x$).

Theorem 2 *Suppose* $0 < r_1 < 1 < r_2$. (i) *If* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous solution of Eq.* (1.6) *then* f *is strictly increasing. If, additionally,* f *has a fixed point then* $f(x) = r_i x$ *when* $x \geq 0$ *and* $f(x) = r_j x$ *when* $x < 0$ *for some* $i, j = 1, 2$. (ii) *Conversely, every continuous solution* $f : \mathbb{R} \to \mathbb{R}$ *of Eq.* (1.6) *without fixed points depends on an arbitrary initial function.*

More precisely, for $x_0 = 0$, for every $x_1 > 0$ (resp. < 0) and for every function $f_0 : [x_0, x_1] \to \mathbb{R}$ (resp. $f_0 : [x_1, x_0] \to \mathbb{R}$) such that

$$f_0(x_0) = f_0(0) = x_1, \quad f_0(x_1) = (r_1 + r_2)x_1, \tag{3.10}$$

$$r_1 \le (f_0(x) - f_0(x'))/(x - x') \le r_2, \quad \forall x, x' \ne 0. \tag{3.11}$$

there exists a unique continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying Eq. (1.6) and $f(x) = f_0(x)$ on $[x_0, x_1]$ (resp. on $[x_1, x_0]$).

Proof By Lemma 3 i) f is strictly increasing. By Lemma 4 the only available fixed point of f is 0. Thus either $0 < f(x) < x$ or $f(x) > x$ for $x > 0$. In the first case $f^n(x) \to 0$ as $n \to \infty$, so (2.6a) in Lemma 2 implies $f(x) = r_1 x$. The second case can be reduced to the first one because $0 < f^{-1}(x) < x$ for $x > 0$, and (2.6b) implies $f^{-1}(x) = x/r_2$, *i.e.*, $f(x) = r_2 x$. The discussion for $x < 0$ is analogous. To prove (ii) we define the sequences $\{x_n\}$ and $\{x_{-n}\}$ by (2.8a) and (2.8b), which tend to ∞ and $-\infty$ respectively as $n \to \infty$, and define recursively homeomorphisms f_n and f_{-n} by (3.7) and the corresponding dual formulas. The discussion is quite similar to that in the proof of Theorem 1.

The case where $0 < r_1 < r_2 < 1$ can be obviously reduced to the case of Theorem 1 by considering the dual equation (1.6d).

Theorem 3 *Suppose $r_1 < r_2 < -1$. (i) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of Eq. (1.6) then f is strictly decreasing with a unique fixed point 0 and satisfies the "two-side" Lipschitzian condition $r_1 \le (f(x) - f(x'))/(x - x') \le r_2$ for $x \ne x'$ in $\mathbb{R}$. (ii) Conversely, Eq. (1.6) has a continuous solution depending on an arbitrary function, given by $f(x) = -p(x)$ when $x \ge 0$ and $f(x) = p(-x)$ when $x < 0$ where $p : [0, \infty) \to [0, \infty)$ has been constructed in Theorem 1 as an arbitrary solution of the functional equation*

$$p^2(x) = ((-r_1) + (-r_2))p(x) - (-r_1)(-r_2)x, \quad x \in [0, \infty). \tag{3.12}$$

Proof Lemma 3 iv) implies all results of (i) except for that $f(0) = 0$. If $f(x_0) > x_0$ (resp. $< x_0$) for some $x_0 \in \mathbb{R}$ then by its "two-side" Lipschitzian condition

$$f(x) \le f(x_0) + r_2(x - x_0) \to -\infty, \quad \text{as} \quad x \to +\infty, \tag{3.13a}$$

resp.

$$f(x) \ge f(x_0) + r_2(x - x_0) \to +\infty, \quad \text{as} \quad x \to -\infty. \tag{3.13b}$$

Thus $f(x_1) < x_0 < x_1$ (resp. $f(x_1) > x_0 > x_1$) for some $x_1 \in \mathbb{R}$. By the continuity f must have a fixed point. By Lemma 4, $f(0) = 0$ and $f(x) \ne x$ for $x \ne 0$. Furthermore, in order to prove (ii) it suffices to check that $f(x)$ defined by $p(x)$ in (3.12) satisfies Eq. (1.6). For $x \ge 0$, $f(f(x)) = f(-p(x)) = p(-(-p(x)) = p^2(x) = (r_1 + r_2)(-p(x)) - r_1 r_2 x = (r_1 + r_2)f(x) - r_1 r_2 x$. Similarly for $x < 0$.

Theorem 4 *Suppose $r_1 < -1 < r_2 < 0$. Then every continuous solution f of Eq. (1.6) is strictly decreasing and 0 is its unique fixed point, and $r_1 \le (f(x) - f(x'))/(x - x') \le r_2, \forall x \ne x'$.*

The proof proceeds as in the case of Theorem 3 (i) in the light Lemmas 3 iv) and 4.

The case where $-1 < r_1 < r_2 < 0$ can be obviously reduced to the case of Theorem 3 by considering the dual equation (1.6d).

4 Noncritical Cases where $r_1 r_2 < 0$

Theorem 5 *Suppose that $r_1 < 0, r_1 \neq -1, r_2 > 0, r_2 \neq 1$ and $r_2 \neq -r_1$. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous solutions of Eq. (1.6) then $f(x) = r_1 x$ or $f(x) = r_2 x$ for $x \in \mathbb{R}$.*

Proof It is discussed according to the following different cases.

Case i) $-1 < r_1 < 0,\ 0 < r_2 < 1$ and $|r_1| < r_2$. In view of Lemma 3 ii) every decreasing solution is of the form $f(x) = r_1 x + c$. We can easily check that $c = 0$ by substituting this function in Eq. (1.6). Alternatively, we prove that $f(x) = r_2 x$ is the unique increasing solution. To prove indirectly, we assume that there is a continuous increasing solution f different from the function $x \mapsto r_2 x$. Since $|r_1| < 1$ and $|r_2| < 1$ in this case, by Lemma 2, $f^n(x) \to 0$ for $x \in \mathbb{R}$ as $n \to \infty$. The monotonicity implies $f(0) = 0$ and Lemma 4 implies that f has no fixed points other than 0. Thus by Lemma 3 ii) we obtain

$$r_2 x < f(x) < x, \quad \text{as } x > 0; \quad x < f(x) < r_2 x, \quad \text{as } x < 0. \tag{4.1}$$

Note that the reason why the inequalities in (4.1) are strict is that $f(x) \neq r_2 x, \forall x \neq 0$; otherwise, if $f(x_0) = r_2 x_0$ for some $x_0 > 0$, for example, then fixing $x_1 = 0$ and $x_2 = x_0$ respectively in (2.15) we have $f(x) \geq r_2 x$ for $x \geq 0$ and $f(x) \leq r_2 x$ for $x \in [0, x_0)$ and thus $f(x) \equiv r_2 x$ for $x \in [0, x_0]$, *i.e.*, $f(x) = r_2 x$ for $x \geq 0$ by the continuous extension and increasing iteration of Eq. (1.6). Therefore, for $x > 0$ the sequence $\{f^n(x)\}$ is strictly decreasing, $\{f^{-n}(x)\}$ is strictly increasing, and

$$r_2^n x < f^n(x) < x, \quad x < f^{-n}(x) < (1/r_2)^n x. \tag{4.2}$$

Take $x_0 > 0$ and put $x_n = f^n(x_0), n = 0, \pm 1, \cdots$. Since $\{x_{-n+1} : n = 0, 1, \cdots\}$ satisfies (2.8b), by (2.9b) and the monotonicity of $\{f^{-n}(x)\}$ we have that $x_{-n+1} < x_{-n}$, *i.e.*,

$$\begin{aligned} &B_2(-n)(x_0 - x_1/r_1) - B_1(-n)(x_0 - x_1/r_2) \\ &< B_2(-n-1)(x_0 - x_1/r_2) - B_1(-n-1)(x_0 - x_1/r_2). \end{aligned} \tag{4.3}$$

Multiplying both sides of (4.3) by the negative constant $(1/r_2 - 1/r_1) r_1 r_2$, we get

$$(1/r_1)^n G > (1/r_2)^n H, \quad i.e., \quad (r_2/r_1)^n G > H \tag{4.4}$$

for $n = 0, 1, \cdots$, where $G = r_2 x_0 - x_1 - r_1 r_2 x_0 + r_1 x_1$ and $H = r_1 x_0 - x_1 - r_1 r_2 x_0 + r_2 x_1$. If $G > 0$,

$$-\infty = \lim_{k \to \infty} (r_2/r_1)^{2k+1} G \geq H, \tag{4.5a}$$

which implies a contradiction; if $G < 0$,

$$G > (r_1/r_2)^{2k} H \to 0, \quad \text{as} \quad k \to \infty, \tag{4.5b}$$

which is also a contradiction. Consequently, $G = 0$, *i.e.*, $f(x_0) = x_1 = r_2 x_0$. This conflicts with (4.1). The proofs are analogous for $x_0 < 0$ and $x < 0$.

Case ii) $-1 < r_1 < 0,\ 0 < r_2 < 1$ and $|r_1| > r_2$. $f(x) = r_1 x$ is the unique decreasing solution, by Lemma 3 iii) as in the proof of the case i). Similarly we assume that Eq. (1.6) has an increasing continuous solution f different from the function $x \mapsto r_2 x$. In the same way as above we obtain strict inequalities

$$0 < f(x) < r_2 x < x, \quad \text{when} \quad x > 0; \quad x < r_2 x < f(x) < 0, \quad \text{when} \quad x < 0. \tag{4.6}$$

It follows that $\{x_n : x_n = f^n(x_0), n = 0, 1, \cdots\}$ satisfies $x_{n+1} < x_n$, and by (2.9a) we have

$$(r_2/r_1)^n H > G, \quad n = 0, 1, \cdots, \tag{4.7}$$

where $G = x_1 - r_2x_0 - r_1x_1 + r_1r_2x_0$ and $H = x_1 - r_1x_0 - r_2x_1 + r_1r_2x_0$. Consequently, as in the case i) we get $G = 0$, *i.e.*, $f(x_0) = x_1 = r_2x$, thus reducing to an absurdity.

Case iii) $r_1 < -1, r_2 > 1$ and $|r_1| < r_2$. This case can be reduced to the case ii) by considering Eq. (1.6d) for f^{-1}.

Case iv) $r_1 < -1, r_2 > 1$ and $|r_1| > r_2$. Similarly this case can be reduced to the case i) as in the case iii).

Case v) $r_1 < -1$ and $0 < r_2 < 1$. Obviously, by Lemma 3 iii) $f(x) = r_1x$ is the unique decreasing solution. Suppose $g(x)$ is an increasing continuous solution. By Lemma 3 iii)

$$0 \le (g(x) - g(y))/(x - y) \le r_2, \quad \forall x \ne y. \tag{4.8}$$

Since $0 < r_2 < 1$ we see by the contraction principle that g has a unique fixed point. By Lemma 4, $g(0) = 0; g(x) \le r_2x < x$ when $x > 0$; $g(x) \ge r_2x > x$ when $x < 0$. Then the monotonicity implies $\{g^n(x)\} \to 0$ as $n \to \infty$. It follows from (2.6a) in Lemma 2 that $g(x) = r_2x$.

Case vi) $-1 < r_1 < 0$ and $r_2 > 1$. This case follows immediately from the case iii) by considering Eq. (1.6d) for f^{-1}. This completes the proof.

Theorem 5 shows that in these cases the possible continuous solutions of Eq. (1.6) are its characteristic solutions.

5 The Case $|r_1| = |r_2|$

Theorem 6 *Suppose $r_1 = r_2 = r \ne 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of Eq.* (1.6). *Then* (i) *if $r \ne 1, f(x) = rx$ for $x \in \mathbb{R}$;* (ii) *if $r = 1, f(x) = x + c$ for $x \in \mathbb{R}$ and for some $c \in \mathbb{R}$.*

Proof By Lemma 2 ii) we have

$$u(x) := \lim_{n\to\infty} (f^n(x)/nr^{n-1}) = f(x) - rx, \quad \forall x \in \mathbb{R}, \tag{5.1a}$$

$$v(x) := \lim_{n\to\infty} (r^{n-1}f^{-n}(x)/n) = f^{-1}(x) - x/r, \quad \forall x \in \mathbb{R}. \tag{5.1b}$$

For $r > 0$, $f(x)$ must be increasing; otherwise, by the monotonicity given in Lemma 1 $f^2(x)$ is increasing and then by putting n even in (5.1a) and (5.1b) $u(x)$ and $v(x)$ are nondecreasing, but for decreasing $f(x)$ the function $u(x) = f(x) - rx$ is clearly decreasing. Since $f(x)$ is increasing, we see from (5.1a) and (5.1b) that $u(x), v(x)$ and $v(f(x))$ are all nondecreasing, *i.e.*,

$$f(x_1) - rx_1 \le f(x_2) - rx_2, \quad x_1 - f(x_1)/r \le x_2 - f(x_2)/r, \tag{5.2}$$

for $x_1 < x_2$. It follows that

$$f(x_2) - f(x_1) = r(x_2 - x_1), \tag{5.3}$$

i.e., for a fixed x_1 and an arbitrary $x, f(x) = rx + c$, where $c = f(x_1) - rx_1$ is a constant. Similarly for $r < 0, f(x)$ must be decreasing and then both $u(x)$ and $v(x)$ are nonincreasing but $v(f(x))$ is nondecreasing. In the same way as above we also get the same form of f.

Furthermore, substituting $f(x) = rx + c$ in (1.6r) we get $c(r-1) = 0$. Thus $c = 0$ when $r \neq 1$. This completes the proof.

Remark 4 Obviously the characteristic roots of Eq. (1.3) are $r_1 = r_2 = 1$. By Theorem 6 all continuous solutions of Eq. (1.2), proposed by Bivens[10], are of the form $f(x) = mx + c$ for some $c \in \mathbb{R}$.

In the remaining case where $r_1 = -r$ and $r_2 = r$ for $r > 0$, Eq. (1.6) is just the problem of iterative roots

$$f^2(x) = r^2 x. \tag{5.4}$$

Kuczma's Theorems 15.7 and 15.9 in Chapter XV of [1] indicate that (5.4) has not only increasing continuous solutions but also decreasing ones, all of which depend on arbitrarily given functions. In particular, when $r = 1$ his Theorem 15.2 shows that Eq. (5.4) has a decreasing solution, the so-called "involutory function" depending on an arbitrary function, but $f(x) = x$ is its unique increasing solution.

6 Critical Cases

In all critical cases there must be a characteristic root with absolute value 1.

Theorem 7 *Suppose $r_2 = 1$ and $0 < r_1 \neq 1$. Then a continuous solution $f : \mathbb{R} \to \mathbb{R}$ of Eq.* (1.6) *has one of the following forms*

$$f(x) = x, x \in \mathbb{R}, \qquad f(x) = \begin{cases} x, & x \le a, \\ r_1 x + (1 - r_1)a, & x > a, \end{cases}$$

$$f(x) = \begin{cases} r_1 x + (1 - r_1)a, & x \le a, \\ x, & x > a, \end{cases} \quad f(x) = \begin{cases} r_1 x + (1 - r_1)a, & x < a, \\ x, & a \le x \le b, \\ r_1 x + (1 - r_1)b, & x > b, \end{cases}$$

where $a, b \in \mathbb{R}$ and $a < b$. Conversely all these functions are solutions of Eq. (1.6).

Proof In the case where $0 < r_1 < 1, F := \{x \in \mathbb{R} : f(x) = x\}$ is a connected closed interval (or single point); otherwise, if $f(x) \neq x$ when $x \in (a, b)$ for some $a, b \in F$ with $a < b$, or more concretely if $f(x) > x$ (resp. $< x$) when $x \in (a, b)$,

$$(f(x) - f(a))/(x - a) > 1 = r_2, \quad x \in (a, b), \tag{6.1a}$$

resp.

$$(f(b) - f(x))/(b - x) > 1 = r_2, \quad x \in (a, b), \tag{6.1b}$$

and then by Lemma 3 i) we get a contradiction. In what follows we discuss F case by case. When $F = \mathbb{R}, f(x) = x$ for $x \in \mathbb{R}$. When $F = (-\infty, a]$, Lemmas 1 and 3 i) and the fact $f(F) = F$ imply that $f(x)$ is strictly increasing from (a, ∞) onto itself. By Lemma 3 i) we have $a < f(x) < x$ for $x \in (a, \infty)$. Hence $f^n(x) \to a$ as $n \to \infty$. It follows from (2.6a) in Lemma 2 that $f(x) = r_1 x + (1 - r_1)a$ for $x > a$. Similar discussions for $F = [a, \infty)$ and $F = [a, b]$ give the desired solutions. The other case where $r_1 > 1$ can be reduced easily to the previous one by considering the dual equation (1.6d).

Theorem 8 *Suppose $r_2 = 1$ and $-1 \neq r_1 < 0$. $f : \mathbb{R} \to \mathbb{R}$ is a continuous solutions of Eq.* (1.6). *Then $f(x) = x$ for $x \in \mathbb{R}$ or $f(x) = r_1 x + c$ for $x \in \mathbb{R}$ where $c \in \mathbb{R}$ is a constant.*

Proof In the case where $-1 < r_1 < 0$, (2.6a) implies

$$g(x) := \lim_{n\to\infty} f^n(x) = (r_2 - r_1)^{-1}(f(x) - r_1 x). \tag{6.2}$$

If f is increasing then g is strictly increasing and continuous from $\mathbb{R}$ onto $\mathbb{R}$. Thus

$$f(g(x)) = f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x) = g(x), \quad x \in \mathbb{R}. \tag{6.3}$$

This means that $f(x) = x$ for $x \in \mathbb{R}$. On the other hand, if f is decreasing then Lemma 3 ii) implies $f(x) = r_1 x + c, x \in \mathbb{R}$, for some $c \in \mathbb{R}$. The other case where $r_1 < -1$ can be reduced to the previous one by considering the dual equation (1.6d).

Theorem 9 *Suppose $r_1 = -1$ and $0 < r_2 \neq 1$. f is a continuous solution of Eq. (1.6). Then $f(x) = -x$ for $x \in \mathbb{R}$ or $f(x) = r_2 x$ for $x \in \mathbb{R}$.*

Proof In the case where $0 < r_2 < 1$, by Lemma 3 iii), $f(x) = -x$ for $x \in \mathbb{R}$ is the unique decreasing solution, and if f is increasing then

$$0 \le (f(x_2) - f(x_1))/(x_2 - x_1) \le r_2 < 1, \quad \forall x_1 \neq x_2. \tag{6.4}$$

In this circumstance f, as a contraction, has a unique fixed point, which must be 0 by Lemma 4. Naturally from (6.4) we have $f(x) < x$ (resp. $> x$) for $x > 0$ (resp. < 0), so $f^n(x) \to 0$ for $x \in \mathbb{R}$ as $n \to \infty$. However, by Lemma 2,

$$f^n(x) = (r_2^n/(r_2+1))(f(x) + x) - ((-1)^n/(r_2+1))(f(x) - r_2 x), \tag{6.5}$$

for $x \in \mathbb{R}$. Therefore$f(x) = r_2 x$ for $x \in \mathbb{R}$. The other case where $r_2 > 1$ can be reduced to the previous one by considering (1.6d).

Theorem 10 *Suppose $r_1 = -1$ and $-1 \neq r_2 < 0$. Then the functions $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = -x, \quad x \in \mathbb{R} \tag{6.6}$$

and

$$f(x) = r_2 x, \quad x \in \mathbb{R} \tag{6.7}$$

and continuous solutions of Eq. (1.6).

Proof Substituting the functions f defined by (6.6) and (6.7) in (1.6r) we can verify directly our result.

7 The Case of No Real Roots

Theorem 11 *Eq.* (1.6) *has no continuous solutions on $\mathbb{R}$ if it has no real characteristic roots.*

Proof Assume Eq. (1.6) has a continuous solution $f : \mathbb{R} \to \mathbb{R}$ and a pair of complex characteristic roots

$$r_1 = a - ib = S\exp(-i\theta), \quad r_2 = a + ib = S\exp(i\theta), \tag{7.1}$$

where $a, b \in \mathbb{R}, b > 0, S > 0$ and $\theta \in (0, \pi)$. By Lemma 1 f is monotone and f^2 is strictly increasing. By Lemma 4 $f(x) \neq x$ for $x \neq 0$. Thus the sign of the sequence $\{f^{n+1}(x) - f^n(x)\}$

is the same (resp. alternates between -1 and 1) for arbitrary fixed $x \neq 0$ when f is strictly increasing (resp. decreasing). However, from (2.6a)

$$\begin{aligned} f^n(x) &= (r_2^n/(r_2-r_1))(f(x)-r_1x)+(r_1^n/(r_2-r_1))(r_2x-f(x)) \\ &= b^{-1}S^n\sin\theta\cdot f(x)-b^{-1}S^{n+1}\sin(n-1)\theta\cdot x. \end{aligned} \tag{7.2}$$

Then

$$f^{n+1}(x)-f^n(x)=r_2^nU(x)+r_1^nV(x), \tag{7.3}$$

where $U(x)=((r_2-1)/(r_2-r_1))(f(x)-r_1x)$ and $V(x)=((r_1-1)/(r_2-r_1))(r_2x-f(x))$. Clearly $\overline{U}(x)=V(x)$, so for a fixed $x\neq 0$ we can let

$$U(x)=T\exp(it) \quad \text{and} \quad V(x)=T\exp(-it), \tag{7.4}$$

where $T\geq 0$ and $t\in[0,2\pi)$. Hence

$$\begin{aligned} f^{n+1}(x)-f^n(x) &= S^nT(\exp(i(n\theta+t))+\exp(-i(n\theta+t))) \\ &= 2S^nT\cos(n\theta+t). \end{aligned} \tag{7.5}$$

Since $S>0$, when $T>0$ (7.5) conflicts with the property of the sign of $\{f^{n+1}(x)-f^n(x)\}$ stated as above; when $T=0$ we see $U(x)=V(x)=0$, *i.e.*, $f(x)=r_1x=r_2x$ for all $x\neq 0$, and thus we get a ridiculous result that $r_1=r_2$. This completes the proof.

Acknowledgements The authors are grateful to Prof. Jiehua Mai (Shantou University) for his suggestions and to referees for helpful comments. The second author owes the colleagues in Katowice and Bielsko-Biala of Poland for their hospitality when he worked there.

References

1 Kuczma M. Functional Equations in a Single Variable, Monografie Matematyczne. Tom 46, Warszwa, 1968
2 Kuczma M. On monotonic solutions of a functional equation (I). Ann Polon Math, 1960, 9: 295–297
3 Kuczma M. On monotonic solutions of a functional equation (II). Ann Polon Math, 1961, 10: 161–166
4 Fort Jr, M K. Continuous solutions of a functional equation. Ann Polon Math, 1963, 13: 205–211
5 Mukherjea A, Ratti J S. On a functional equation involving iterates of a bijection on the unit interval. Nonlinear Analysis, 1983, 7: 899–908
6 Zhang Weinian. Discussion on the solutions of the iterated equation $\sum_{i=1}^{n}\lambda_i f^i(x)=F(x)$. Chin Sci Bul, 1987, 32(21): 1444–1451
7 Zhang Weinian. Stability of solutions of the iterated equation $\sum_{i=1}^{n}\lambda_i f^i(x)=F(x)$. Acta Math Sci, 1988, 8(4): 421–424
8 Zhang Weinian. Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n}\lambda_i f^i(x)=F(x)$. Nonlinear Analysis, 1990, 15(4): 387–398
9 Nabeya S. On the functional equation $f(p+qx+rf(x))=a+bx+cf(x)$. Aequationes Math, 1974, 11: 199–211
10 Bivens I C. Problem. Amer Math Monthly, 1984, 91(7): 938