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On the Composition of Homogeneous Quasi-Arithmetic Means

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Let ϕ, ψ, γ, $\beta:(0,\infty)\to\mathbf{R}$, strictly monotonic and continuous functions, be the generators of the positively homogeneous quasi-arithmetic means M_ϕ, M_ψ, M_γ, and M_β. The main result gives full characterizations of the functions ϕ, ψ, γ, and β such that

$$M_\phi\big(M_\psi(x,y), M_\gamma(x,y)\big) = M_\beta(x,y), \qquad x,y>0.$$

INTRODUCTION

A mean on $(0,\infty)$ is a function $M:(0,\infty)^2\to(0,\infty)$ having the weak internal property

$$\min\{x,y\} \le M(x,y) \le \max\{x,y\}, \qquad x,y>0.$$

If M, N, and K are means on $(0,\infty)$ then, obviously, the function

$$(x,y)\to M(N(x,y),K(x,y)),$$

the *composition of the means* M, N, and K, is again a mean on $(0,\infty)$. Moreover, if M, N, and K are positively homogeneous then so is their composition.

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A special role plays the class of quasi-arithmetic means. Recall that a mean M is called *quasi-arithmetic* if there exists a strictly monotonic and continuous function $\phi:(0,\infty)\to\mathbf{R}$, a *generator* of the mean, such that $M=M_\phi$, where

$$M_\phi(x,y)=\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right),\qquad x,y>0.$$

It is easy to verify that in general the composition of three quasi-arithmetic means is not a quasi-arithmetic mean. The main result of this paper gives a complete characterization of the positively homogeneous quasi-arithmetic means M_ϕ, M_ψ, M_γ, M_β, which satisfy the composition equation

$$M_\phi\big(M_\psi(x,y),M_\gamma(x,y)\big)=M_\beta(x,y)$$

for all $x,y>0$. As a corollary we obtain the relations

$$G(A(x,y),H(x,y))=G(x,y),$$

$$A(A(x,y),G(x,y))=\big(A(x^{1/2},y^{1/2})\big)^2,$$

for all $x,y>0$, where A, G, and H stand, respectively, for the arithmetic, geometric, and harmonic mean. It turns out that these relations are, in a sense, exceptional, and play a basic role, as we show that the quasi-arithmetic means satisfying the composition equation can be determined from the identities

$$G\Big(\big(A(x^p,y^p)\big)^{1/p},\big(H(x^p,y^p)\big)^{1/p}\Big)=G(x,y),\quad p\in\mathbf{R}\setminus\{0\},\,x,y>0;$$

$$A\big(A(x^p,y^p),G(x^p,y^p)\big)^{1/p}$$
$$=\big(A(x^{p/2},y^{p/2})\big)^{2/p},\quad p\in\mathbf{R}\setminus\{0\},\,x,y>0.$$

Since for any two means M, $N:(0,\infty)^2\to(0,\infty)$ we have

$$M(N(x,y),N(x,y))=N(x,y),\qquad x,y>0,$$

the composition equation is trivially satisfied if $M_\psi=M_\gamma=M_\beta$.

1. PRELIMINARIES

A function $M:(0,\infty)^2\to(0,\infty)$ satisfying the inequality

$$\min\{x,y\}\le M(x,y)\le\max\{x,y\},\qquad x,y>0,$$

is said to be a mean on $(0,\infty)$. It follows that every mean has the property

$$M(x,x)=x,\qquad x>0. \tag{1}$$

A mean M is *positively homogeneous* if

$$M(tx, ty) = tM(x, y), \qquad t, x, y > 0.$$

Let $\phi : (0, \infty) \to \mathbf{R}$ be a continuous and strictly monotonic function. Then it is easy to see that the function $M_\phi : (0, \infty)^2 \to (0, \infty)$ defined by

$$M_\phi(x, y) = \phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right), \qquad x, y > 0, \tag{2}$$

is a mean, and it is called *quasi-arithmetic* (cf. [1, p. 279; 2, p. 245]). The function ϕ will be called a *generator* of the mean M_ϕ. It is well known that the quasi-arithmetic mean is positively homogeneous iff it coincides with a *power mean*.

In this paper the one-parameter family of *power mean* $\mathbf{m}_p : (0, \infty)^2 \to (0, \infty)$ defined by

$$\mathbf{m}_p(x, y) := \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{1/p}, & p \neq 0 \\ \sqrt{xy}, & p = 0 \end{cases} \qquad x, y > 0,$$

plays a key role. Let us note some of the most important properties of this family of means.

Property 1. For every $p \in \mathbf{R}$, $\mathbf{m}_p$ is a quasi-arithmetic mean. For every $p \neq 0$, the function

$$\phi(x) = ax^p + b, \qquad x > 0,$$

with arbitrarily fixed $a, b \in \mathbf{R}$, $a \neq 0$, is a generator of $\mathbf{m}_p$, and

$$\phi(x) = a \log x + b, \qquad x > 0,$$

with arbitrarily fixed $a, b \in \mathbf{R}$, $a \neq 0$, is a generator of $\mathbf{m}_0$.

Property 2. For every $p \in \mathbf{R}$, $\mathbf{m}_p$ is positively homogeneous.

Property 3. The function $\mathbf{R} \ni p \to \mathbf{m}_p$ is continuous and strictly increasing.

Property 4. For every $p \in \mathbf{R}$, $\mathbf{m}_p$ is strictly internal, i.e.,

$$\min\{x, y\} < \mathbf{m}_p(x, y) < \max\{x, y\}, \qquad x, y > 0, x \neq y,$$

In the sequel we denote by A, G, and H, respectively, the arithmetic, geometric, and harmonic means. Note that

$$\mathbf{m}_1 = A, \qquad \mathbf{m}_0 = G, \qquad \mathbf{m}_{-1} = H.$$

2. MAIN RESULT ABOUT COMPOSITIONS OF POWER MEANS

In this section we prove the following

THEOREM 1. *Let $p, q, r, s \in \mathbf{R}$. Then*

$$\mathbf{m}_p\big(\mathbf{m}_q(x,y), \mathbf{m}_r(x,y)\big) = \mathbf{m}_s(x,y), \qquad x, y > 0. \tag{3}$$

if, and only if, one of the following cases occurs:

(1°) $q = r = s$, *and* $p \in \mathbf{R}$ *is arbitrary*;

(2°) $q = p$, $r = 0$, $s = p/2$, *and* $p \in \mathbf{R}$ *is arbitrary*;

(3°) $r = p$, $q = 0$, $s = p/2$, *and* $p \in \mathbf{R}$ *is arbitrary*;

(4°) $p = 0 = s$, $q + r = 0$.

Proof. It follows from the positive homogeneity of the power means that Eq. (3) is equivalent to

$$\mathbf{m}_p\big(\mathbf{m}_q(x,1), \mathbf{m}_r(x,1)\big) = \mathbf{m}_s(x,1), \qquad x > 0. \tag{4}$$

Suppose first that this relation holds true for some numbers $p, q, r, s \in \mathbf{R}$, all different from 0, and such that $q \neq r$, i.e., that

$$\left(\frac{x^q+1}{2}\right)^{p/q} + \left(\frac{x^r+1}{2}\right)^{p/r} = 2\left(\frac{x^s+1}{2}\right)^{p/s}, \qquad x > 0. \tag{5}$$

Without any loss of generality we can assume that $q < r$. From (4), by the definition of mean, we have

$$\min\{\mathbf{m}_q(x,1), \mathbf{m}_r(x,1)\} \le \mathbf{m}_s(x,1) \le \max\{\mathbf{m}_q(x,1), \mathbf{m}_r(x,1)\}, \quad x > 0.$$

By Property 3, $q \le s \le r$, and

$$\mathbf{m}_q(x,1) \le \mathbf{m}_s(x,1) \le \mathbf{m}_r(x,1), \qquad x > 0.$$

If $s = q$, then we would have $\mathbf{m}_s = \mathbf{m}_q$. Hence, in view of Property 4, and (4), we infer that $\mathbf{m}_q = \mathbf{m}_s = \mathbf{m}_r$, and consequently $q = r$, which is a contradiction. In the same way we show that $s = r$ implies $q = r$. This discussion proves that $q < s < r$. Taking the derivatives of both sides of (5) gives

$$x^{q-s}\left(\frac{x^q+1}{2}\right)^{(p-q)/q} + x^{r-s}\left(\frac{x^r+1}{2}\right)^{(p-r)/r} = 2\left(\frac{x^s+1}{2}\right)^{(p-s)/s}, \qquad x > 0.$$

Note that

$$\text{if } s > 0 \text{ then } \lim_{x \to 0+} 2\left(\frac{x^s + 1}{2}\right)^{(p-s)/s} = 2^{2-p/s},$$

and

$$\text{if } s < 0 \text{ then } \lim_{x \to \infty} 2\left(\frac{x^s + 1}{2}\right)^{(p-s)/s} = 2^{2-p/s}.$$

On the other hand, as $q - s < 0 < r - s$, we have

$$\lim_{x \to 0+} x^{q-s} = \infty, \qquad \text{and} \qquad \lim_{x \to \infty} x^{r-s} = \infty.$$

The above relation implies that the limits

$$\lim_{x \to 0+} x^{q-s}\left(\frac{x^q + 1}{2}\right)^{(p-q)/q}, \qquad \lim_{x \to 0+} x^{r-s}\left(\frac{x^r + 1}{2}\right)^{(p-r)/r},$$

as well as

$$\lim_{x \to \infty} x^{q-s}\left(\frac{x^q + 1}{2}\right)^{(p-q)/q}, \qquad \lim_{x \to \infty} x^{r-s}\left(\frac{x^r + 1}{2}\right)^{(p-r)/r},$$

must be finite, and at least one of them is positive.

Suppose for instance that the first of these limits is positive. It follows that $q - s = q - p$, i.e., $s = p$. Consequently $q - p < 0 < r - p$, and

$$x^{q-p}\left(\frac{x^q + 1}{2}\right)^{(p-q)/q} + x^{r-p}\left(\frac{x^r + 1}{2}\right)^{(p-r)/r} = 2, \qquad x > 0.$$

Taking the first and the second derivative of both sides and then setting $x = 1$ gives, respectively, $p = (q + r)/2$, and

$$2p^3 + 12p^2 - p(3q^2 + 6q + 3r^2 + 6r - 16)$$
$$+ 2(q^3 - 4q + r(r^2 - 4)) = 0.$$

Eliminating p from these relations easily gives

$$(q - r)^2(q + r) = 0,$$

i.e., either $r = -q$, and consequently $p = 0$, or $r = q$, which is a desired contradiction. If we assume that one of the remaining three limits is

positive then a similar argument gives a contradiction. Thus we have shown that if $p, q, r, s \in \mathbf{R}$, all different from 0, satisfy (3) then $q = r = s$.

Conversely, it is easy to see that for all $p, q, r, s \in \mathbf{R}$ such that $q = r = s$, Eq. (3) is fulfilled.

Assume now that, in relation (4), exactly one of the numbers p, q, r, s is equal to 0.

First consider the case $p = 0$ and $q \neq 0$, $r \neq 0$, $s \neq 0$. From (4), by the definition of a mean, we have either $q \leq s \leq r$ or $r \leq s \leq q$. If $q = s$ or $r = s$, then, making use of Properties 3 and 4, we would have $q = s$, and, consequently, $q = r = s$. Since in this case relation (3) holds true, we can assume that either $q < s < r$ or $r < s < q$. As the roles of q and r are symmetric, it is enough to consider the case

$$q < s < r.$$

We can write (4) in the form

$$\left(\frac{x^q + 1}{2}\right)^{1/q} \left(\frac{x^r + 1}{2}\right)^{1/r} = \left(\frac{x^s + 1}{2}\right)^{2/s}, \qquad x > 0.$$

Differentiating both sides of this equation gives

$$x^{q-s}\mathbf{m}_r(x,1)\left(\frac{x^q + 1}{2}\right)^{(1-q)/q} + x^{r-s}\mathbf{m}_q(x,1)\left(\frac{x^r + 1}{2}\right)^{(1-r)/r}$$
$$= 2\left(\frac{x^s + 1}{2}\right)^{(2-s)/s},$$

for all $x > 0$. If $s > 0$, and $x \to 0$, then the right hand side tends to $2^{(2s-2)/s}$; similarly, if $s < 0$, and $x \to \infty$. Since

$$q - s < 0 < r - s,$$

if $x \to 0$ or $x \to \infty$ then the left hand side tends either to 0 or to ∞. This is a contradiction.

Now consider the case $r = 0$ and $p \neq 0$, $q \neq 0$, $s \neq 0$. By the definition of the power means we can write (4) in the form

$$\left(\frac{x^q + 1}{2}\right)^{p/q} + x^{p/2} = 2\left(\frac{x^s + 1}{2}\right)^{p/s}, \qquad x > 0.$$

Replacing here x by $x^{2/p}$ we have

$$\left(\frac{x^{2q/p}+1}{2}\right)^{p/q}+x=2\left(\frac{x^{2s/p}+1}{2}\right)^{p/s}, \qquad x>0. \tag{6}$$

Dividing both sides by x^2 and letting $x \to \infty$, gives $2^{-p/q}=2^{(1-p)/s}$, and, consequently,

$$s=\frac{pq}{p+q}, \qquad \text{and} \qquad p+q \neq 0. \tag{7}$$

Hence, taking the first derivative of both sides of (6), for all $x>0$ we get

$$\left(\frac{x^{2q/p}+1}{2}\right)^{(p-q)/q} x^{(2q-p)/p}+1=2\left(\frac{x^{2q/(p+q)}+1}{2}\right)^{p/q} x^{(q-p)/(p+q)}. \tag{8}$$

Now we prove that $p=q$. For an indirect argument suppose that this equation holds true for some $p, q \in \mathbf{R}$, $p \neq 0 \neq q$, $p+q \neq 0$, and consider the following subcases.

(1) $0<p<q$. Letting $x \to 0$ gives 1 on the left and 0 on the right hand side of (8).

(2) $0<q<p$ and $2q>p$. Letting $x \to 0$ gives 1 on the left and ∞ on the right hand side of (8).

(3) $0<q$ and $2q=p$. Here we can write relation (8) in the form

$$\frac{x+1}{2}+1=2\left(\frac{x^{2/3}+1}{2}\right)^2 x^{1/3}, \qquad x>0,$$

and of course it is false.

(4) $0<q$ and $2q<p$. Letting $x \to \infty$ gives 1 on the left and 0 on the right hand side of (8).

(5) $p<0<q$ and $p+q>0$. Dividing both sides of (8) by $x^{(q-p)/(p+q)}$, for all $x>0$ we get

$$\left(\frac{x^{2q/p}+1}{2}\right)^{(p-q)/q} x^{2q^2/(p(p+q))}+x^{(q+p)/(p-q)}=2\left(\frac{x^{2q/(p+q)}+1}{2}\right)^{p/q}. \tag{9}$$

Letting here $x \to 0$ gives ∞ on the left, and $2^{(1-p)/q}$ on the right.

(6) $p < 0 < q$ and $p + q < 0$. Letting $x \to \infty$ in (9) gives ∞ on the left and $2^{1-p/q}$ on the right.

Observe that the relations (8) and (9) remain valid on replacing p and q by $(-p)$ and $(-q)$. It follows that all the remaining possible subcases can be reduced to the above already considered.

The above discussion shows that relation (6) implies $q = p$. From (7) we have $s = p/2$, and, by the assumption, $r = 0$.

On the other hand, making use of the definition of the family $(\mathbf{m}_p)$, it is easy to verify that, for every $p \in \mathbf{R}$, the numbers

$$p, \qquad q := p, \qquad r := 0, \qquad s := \frac{p}{2}$$

satisfy Eq. (3).

Since the role of q and r is symmetric we can omit analogous considerations in the case $q = 0$ and $p, r, s \neq 0$.

If $s = 0$ and $p \neq 0$, $q \neq 0$, $r \neq 0$, in (4), then either $q < 0 < r$ or $r < 0 < q$. It is easy to check that relation (4), having the form

$$\left(\frac{x^q + 1}{2}\right)^{p/q} + \left(\frac{x^r + 1}{2}\right)^{p/r} = 2x^{p/2}, \qquad x > 0,$$

cannot occur.

Now assume that exactly two of the numbers p, q, r, s are equal to 0. Suppose first that $p = 0 = q$, $r \neq 0 \neq s$. Properties 3 and 4 imply that $s \in (0, r)$ if $r > 0$, and $s \in (r, 0)$ if $r < 0$. Suppose first that $r > 0$. From (4) we have

$$\sqrt{x}\left(\frac{x^r + 1}{2}\right)^{1/r} = \left(\frac{x^s + 1}{2}\right)^{2/s}, \qquad x > 0.$$

Letting $x \to 0$ gives a contradictory relation $0 = 2^{-2/s}$. If $r < 0$ then $s < 0$, and setting $r := m$, $s := -n$, $m, n > 0$, allows us to write the above equation in the form

$$\frac{2^{1/m}}{(x^m + 1)^{1/m}} = \frac{2^{1/n}}{(x^n + 1)^{1/n}}\sqrt{x}\,.$$

Letting $x \to 0$ gives $2^{1/m} = 0$, which is a contradiction.

The same argument shows that if $p = 0 = r$ then there are no real numbers q and s, $q \neq 0 \neq s$, such that (3) is satisfied.

Assume that $p = 0 = s$, and $q \neq 0 \neq r$. From (4) we have

$$\left(\frac{x^q + 1}{2}\right)^{1/q}\left(\frac{x^r + 1}{2}\right)^{1/r} = x, \qquad x > 0.$$

The internality of the mean $\mathbf{m}_p$ and its increasing monotonicity with respect to p imply that either $q < 0 < r$ or $r < 0 < q$. Put $q := -m$. Then $m, r > 0$ and, with some simple calculations, we can write the above relation in the equivalent form

$$2^{1/m}(x^r + 1)^{1/r} = 2^{1/r}(x^m + 1)^{1/m}, \qquad x > 0.$$

Letting $x \to 0$ gives $2^{1/m} = 2^{1/r}$. Thus $r = m$ and, consequently, $r = -q$.

Conversely, taking $p = 0 = s$, arbitrary $q \in \mathbf{R}$, and $r := -q$, we have for all $x, y > 0$

$$\mathbf{m}_p\big(\mathbf{m}_q(x, y), \mathbf{m}_r(x, y)\big) = \mathbf{m}_0\big(\mathbf{m}_q(x, y), \mathbf{m}_{-q}(x, y)\big)$$

$$\left(\left(\frac{x^q + y^q}{2}\right)^{1/q}\left(\frac{x^{-q} + y^{-q}}{2}\right)^{-1/q}\right)^{1/2} = \left(\left(\frac{x^q + y^q}{2}\,\frac{2x^q y^q}{x^q + y^q}\right)^{1/q}\right)^{1/2}$$

$$= \sqrt{xy} = \mathbf{m}_s(x, y).$$

With respect to the symmetrical role of q and r, in the same way we can show that the numbers $p = 0 = s$, and $q < 0 < r$ satisfy (3) if, and only if, $r = -q$.

Assume that $p, q, r, s \in \mathbf{R}$ are such that $q = 0 = r$, $p \neq 0 \neq s$. If they satisfy (4) then

$$\sqrt{x} = \left(\frac{x^s + 1}{2}\right)^{1/s}, \qquad x > 0,$$

which is a contradiction.

Now assume that $p, q, r, s \in \mathbf{R}$, $r = 0 = s$, $p \neq 0 \neq q$. Then (4) reduces to the contradictory relation

$$\left(\frac{x^q + 1}{2}\right)^{p/q} = x^{p/2}, \qquad x > 0.$$

Since it is easy to see that relation (4) is false if exactly three of the numbers p, q, r, s are equal to 0, the proof is completed. ∎

Applying Theorem 1(4°) with $p = 0 = s$, $r = -q$, where $q \neq 0$ is arbitrary, and Theorem 1(2°) with $q = p$, $r = 0$, $s = p/2$, where $p \neq 0$ is

arbitrary, we obtain the following

COROLLARY 1. *For all $p, q \in \mathbf{R} \setminus \{0\}$ and for all $x, y > 0$*

$$\left(\left(\frac{x^q + y^q}{2}\right)^{1/q}\left(\frac{x^{-q} + y^{-q}}{2}\right)^{-1/q}\right)^{1/2} = \sqrt{xy}, \tag{10}$$

$$\left(\frac{\left(\left((x^p + y^p)/2\right)^{1/p}\right)^p + \left(\sqrt{xy}\right)^p}{2}\right)^{1/p} = \left(\frac{x^{p/2} + y^{p/2}}{2}\right)^{2/p}. \tag{11}$$

Remark 1. Note that the relations

$$G(A(x,y), H(x,y)) = G(x,y), \qquad x, y > 0; \tag{12}$$

$$A(A(x,y), G(x,y)) = \mathbf{m}_{1/2}(x,y), \qquad x, y > 0, \tag{13}$$

play here a fundamental role. They are equivalent to the relations mentioned in the statements (2°), (3°), and (4°) of Theorem 1. To get for instance (10) it is enough to replace x and y, respectively, by x^q and y^q in (12), and raise both sides to the power $1/q$. Similarly, replacing, respectively, x and y by x^p and y^p in (13), and then raising it to the power $1/p$, gives relation (11).

3. NOTES ON COMPOSITIONS OF HOMOGENEOUS QUASI-ARITHMETIC MEANS

We need the following (cf. J. Aczél [1, 3.1.2, Theorem 2, p. 153])

LEMMA 1. *Let $\phi : (0, \infty) \to \mathbf{R}$ be a continuous and strictly monotonic function. Then M_ϕ is positively homogeneous if, and only if, either there exist $a, p \in \mathbf{R} \setminus \{0\}$ and $b \in \mathbf{R}$ such that $\phi(x) = ax^p + b$, $x > 0$, or there exist a $b \in \mathbf{R}$, $a \neq 0$, such that $\phi(x) = a \log(x) + b$, $x > 0$.*

Now we can prove the following

THEOREM 2. *Let $\phi, \psi, \gamma, \beta : (0, \infty) \to \mathbf{R}$ be strictly monotonic, and continuous. Suppose that M_ϕ and at least two of the means M_ψ, M_γ, M_β are positively homogeneous. Then*

$$M_\phi\big(M_\psi(x,y), M_\gamma(x,y)\big) = M_\beta(x,y), \qquad x, y > 0, \tag{14}$$

if, and only if, one of the following cases occurs:

(1°) *there exist* $p, q \in \mathbf{R} \setminus \{0\}$, *such that*

$$\phi(x) = a_1 x^p + b_1, \qquad \psi(x) = a_2 x^q + b_2,$$
$$\gamma(x) = a_3 x^q + b_3, \qquad \beta(x) = a_4 x^q + b_4,$$

(2°) *there exists* $p \in \mathbf{R} \setminus \{0\}$ *such that*

$$\phi(x) = a_1 x^p + b_1, \qquad \psi(x) = a_2 x^p + b_2,$$
$$\gamma(x) = a_3 \log(x) + b_3, \qquad \beta(x) = a_4 x^{p/2} + b_4,$$

(3°) *there exists* $p \in \mathbf{R} \setminus \{0\}$ *such that*

$$\phi(x) = a_1 x^p + b_1, \qquad \psi(x) = a_2 \log(x) + b_2,$$
$$\gamma(x) = a_3 x^p + b_3, \qquad \beta(x) = a_4 x^{p/2} + b_4,$$

(4°) *there exists* $q \in \mathbf{R} \setminus \{0\}$ *such that*

$$\phi(x) = a_1 \log(x) + b_1, \qquad \psi(x) = a_2 x^q + b_2,$$
$$\gamma(x) = a_3 x^{-q} + b_3, \qquad \beta(x) = a_4 \log(x) + b_4,$$

for some $a_i, b_i \in \mathbf{R}$, $a_i \neq 0$ $(i = 1, 2, 3, 4)$, *and all* $x > 0$.

Proof. By assumption M_ϕ is positively homogeneous. First we show that all the means M_ψ, M_γ, M_β are positively homogeneous. To this end it is enough to consider four cases.

If M_ψ, M_γ are positively homogeneous, then by (14) so is M_β.

If M_ψ, M_β are positively homogeneous then for all $t, x, y > 0$,

$$M_\beta(tx, ty) = M_\phi\big(M_\psi(tx, ty), M_\gamma(tx, ty)\big) = M_\phi\big(tM_\psi(x, y), M_\gamma(tx, ty)\big)$$

and

$$tM_\beta(x, y) = tM_\phi\big(M_\psi(x, y), M_\gamma(x, y)\big) = M_\phi\big(tM_\psi(x, y), tM_\gamma(x, y)\big).$$

The positive homogeneity of M_β implies that

$$M_\phi\big(tM_\psi(x, y), M_\gamma(tx, ty)\big) = M_\phi\big(tM_\psi(x, y), tM_\gamma(x, y)\big), \qquad t, x, y > 0.$$

Since every quasi-arithmetic mean is strictly increasing with respect to each variable, it follows that $M_\gamma(tx, ty) = tM_\gamma(x, y)$ for all $t, x, y > 0$.

In the same way we can show that if M_γ, M_β are positively homogeneous, then so is M_ψ. Now the result is a consequence of Lemma 1 and Theorem 1. ∎

Remark 2. Suppose that M_ψ, M_γ, M_β are positively homogeneous. Then for all $t, x, y > 0$ we have

$$M_\beta(tx, ty) = M_\phi\big(M_\psi(tx, ty), M_\gamma(tx, ty)\big) = M_\phi\big(tM_\psi(x, y), tM_\gamma(x, y)\big),$$

and

$$tM_\beta(x, y) = tM_\phi\big(M_\psi(x, y), M_\gamma(x, y)\big).$$

From the positive homogeneity of M_β we get

$$M_\phi\big(tM_\psi(x, y), tM_\gamma(x, y)\big) = tM_\phi\big(M_\psi(x, y), M_\gamma(x, y)\big), \qquad t, x, y > 0.$$

Let $u, v > 0$ be such that the system of equations

$$M_\psi(x, y) = u, \qquad M_\gamma(x, y) = v$$

has a solution $x, y > 0$. Then we have

$$M_\phi(tu, tv) = tM_\phi(u, v), \qquad \text{for all } t > 0.$$

To show that, in general, relation (14) and the positive homogeneity of M_ψ, M_γ, M_β do not yield the homogeneity of M_ϕ, consider the following

EXAMPLE 1. Let $\phi : (0, \infty) \to \mathbf{R}$ be an arbitrary non-power monotonic and continuous function, and $\psi, \gamma, \beta : (0, \infty) \to \mathbf{R}$, $\psi(x) = \gamma(x) = \beta(x) = x$, $x > 0$. Then

$$M_\psi(x, y) = M_\gamma(x, y) = M_\beta(x, y) = \frac{x + y}{2}, \qquad x, y > 0,$$

are positively homogeneous, and M_ϕ is not. However, we have

$$M_\phi\big(M_\psi(x, y), M_\gamma(x, y)\big) = M_\phi\left(\frac{x + y}{2}, \frac{x + y}{2}\right) = \frac{x + y}{2} = M_\beta(x, y),$$

for all $x, y > 0$.

In connection with the first statement in Theorem 1 note the following obvious

Remark 3. Let $\phi, \psi, \gamma, \beta : (0, \infty) \to \mathbf{R}$ be strictly monotonic, continuous, and such that $\gamma(x) = a\psi(x) + b$, and $\beta(x) = c\psi(x) + d$, $x > 0$, for some $a, b, c, d \in \mathbf{R}$. Then, by (1),

$$M_\phi\big(M_\psi(x, y), M_\gamma(x, y)\big) = M_\beta(x, y), \qquad x, y > 0.$$

Remark 4. Let $I \subset \mathbf{R}$ be an interval. It is known (cf. J. Aczél and J. Dhombres [2, p. 291]), that a continuous and strictly monotonic in each variable function $M : I^2 \to I$ such that

$$M(x,x) = x, \qquad M(x,y) = M(y,x), \qquad x,y \in I,$$

satisfies the bisymmetry functional equation

$$M[M(x,y), M(z,w)] = M[M(x,z), M(y,w)], \qquad x,y,z,w \in I,$$

if, and only if, M is a quasi-arithmetic mean, i.e.,

$$M(x,y) = \phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right), \qquad x,y \in I,$$

where $\phi : I \to \mathbf{R}$ is a continuous and strictly monotonic function.

Note that this result permits us to determine all continuous and strictly monotonic functions $M, N, K : I^2 \to I$ such that

$$M(x,x) = N(x,x) = K(x,x) = x, \qquad x \in I, \tag{15}$$

$$N(x,y) = N(y,x), \qquad K(x,y) = K(y,x), \qquad x,y \in I, \tag{16}$$

and satisfying the functional equation

$$M[N(x,y), K(z,w)] = M[N(x,z), K(y,w)], \qquad x,y,z,w \in I. \tag{17}$$

To show it first observe that M is symmetric, i.e.,

$$M(x,y) = M(y,x), \qquad x,y \in I.$$

In fact, applying in turn (15), (17), (16), (17), and (15) we obtain

$$\begin{aligned} M(x,y) &= M[N(x,x), K(y,y)] = M[N(x,y), K(x,y)] \\ &= M[N(y,x), K(y,x)] = M[N(y,y), K(x,x)] = M(y,x), \end{aligned}$$

for all $x, y \in I$. From using (17), (16), and again (17), we have

$$\begin{aligned} M[N(x,y), K(z,w)] &= M[N(x,z), K(y,w)] = M[N(z,x), K(w,y)] \\ &= M[N(z,w), K(x,y)] \end{aligned}$$

for all $x, y, z, w \in I$. Setting $w := z$ in this relation gives

$$M[N(x,y), z] = M[z, K(x,y)], \qquad x,y,z \in I,$$

and by the symmetry of M we get

$$M[N(x,y),z] = M[K(x,y),z], \qquad x,y,z \in I.$$

The strict monotonicity of M implies that $N = K$. Now from (17) we have

$$M[K(x,y),K(z,w)] = M[K(x,z),K(y,w)], \qquad x,y,z,w \in I.$$

Setting $z := x$, $w := y$ gives

$$M[K(x,y),K(x,y)] = M[K(x,x),K(y,y)], \qquad x,y \in I,$$

which, in view of (15), means that $K = M$. Thus the Pexider type equation (17) reduces to the bisymmetry equation. (A more general functional equation than (17) was considered by J. Aczél and Gy. Maksa, cf. [3].)

A weaker form of the Pexider bisymmetry equation (17) is the functional equation

$$M[N(x,y),K(z,x)] = M[N(x,z),K(y,z)], \qquad x,y,z \in I, \quad (18)$$

where M, N, $K : I^2 \to I$ are the unknown functions.

We shall prove the following

Remark 5. Let $I \subset \mathbf{R}$ be an interval.

(1°) Suppose that M, N, $K : I^2 \to I$ satisfy Eq. (18). If M is symmetric, injective with respect to the first variable, and

$$N(x,x) = x = K(x,x), \qquad x \in I, \quad (19)$$

then

$$N(x,y) = K(y,x), \qquad x,y \in I.$$

If moreover N or K is symmetric then $N = K$.

(2°) If the functions M, N, $K : I^2 \to I$ are such that M and N are symmetric and $K = N$, then Eq. (18) is fulfilled.

Proof. (1°). Setting $z := x$ in (18), and making use of (19) gives

$$M[N(x,y),x] = M[x,K(y,x)], \qquad x,y \in I.$$

The symmetry of M implies that

$$M[N(x,y),x] = M[K(y,x),x], \qquad x,y \in I.$$

Hence, by the strict monotonicity of M (with respect to the first variable) we obtain $N(x,y) = K(y,x)$ for all $x,y \in I$. Hence, if N or K is symmetric then $N = K$.

The proof of (2°) is obvious. ∎

Setting $z := x$ and $w := y$ in (17) gives the functional equation

$$M[N(x,y),K(x,y)] = M[N(x,x),K(y,y)] = M(x,y), \qquad x,y \in I.$$

Assuming condition (15) is fulfilled we get the functional equation

$$M[N(x,y),K(x,y)] = M(x,y), \qquad x,y \in I. \tag{20}$$

Suppose that $M, N, K : (0,\infty) \to (0,\infty)$ are homogeneous quasi-arithmetic means. In particular, in this paper we have shown that M, N, and K satisfy Eq. (20) if, and only if, there is a $q \in \mathbf{R}$ such that

$$M = G, \qquad N = \mathbf{m}_q, \qquad K = \mathbf{m}_{-q}.$$

In this connection let us note the following

COROLLARY 2. *Let $M, N : (0,\infty)^2 \to (0,\infty)$ be arbitrary means on $(0,\infty)$, and $f : (0,\infty) \to \mathbf{R}$ a function. Then $K : (0,\infty)^2 \to (0,\infty)$ defined by*

$$K(x,y) := f(M(x,y) \cdot N(x,y)), \tag{21}$$

is a mean on $(0,\infty)$ if, and only if, $f(x) = \sqrt{x}$ for all $x > 0$, and, consequently

$$K(x,y) = G(M(x,y) \cdot N(x,y)), \qquad x,y > 0. \tag{22}$$

Suppose that M and N are positively homogeneous and quasi-arithmetic means. Then, apart from the trivial case $M = N(= K)$, K given by (21) is quasi-arithmetic if, and only if, there exists a $q \in \mathbf{R}$ such that

$$M = \mathbf{m}_q \qquad \textit{and} \qquad N = \mathbf{m}_{-q}.$$

Proof. Suppose that K is a mean on $(0,\infty)$. Setting $y = x$ in (21) gives

$$x = K(x,x) = f(M(x,x) \cdot N(x,x)) = f(x^2), \qquad x > 0.$$

Thus $f(x) = \sqrt{x}$, for all $x > 0$, and consequently, (22) holds true. The converse implication is obvious.

Now suppose that M and N are positively homogeneous and quasi-arithmetic. Then K is positively homogeneous. If K is quasi-arithmetic, then M, N, and K must be some power means. Therefore there exist q, r, $s \in \mathbf{R}$ such that

$$M = \mathbf{m}_q, \qquad N = \mathbf{m}_r, \qquad K = \mathbf{m}_s,$$

and, since $G = \mathbf{m}_0$, from (22) we have

$$\mathbf{m}_s(x,y) = \mathbf{m}_0\big(\mathbf{m}_q(x,y), \mathbf{m}_r(x,y)\big), \qquad x,y > 0.$$

Now the result follows from Theorem 1(4°).

Remark 6. Suppose that M and N are means on $(0,\infty)$, and $g:(0,\infty) \to (0,\infty)$ is an arbitrary function. It is easy to verify that if the function $K:(0,\infty)^2 \to (0,\infty)$,

$$K(x,y) := \frac{g(M(x,y))}{N(x,y)}, \qquad x,y>0,$$

is a mean, then $g(x) = x^2$, $x > 0$, and consequently,

$$K(x,y) = \frac{(M(x,y))^2}{N(x,y)}, \qquad x,y>0.$$

Note that Corollary 2 answers the question when the function M^2/N is a positively homogeneous and quasi-arithmetic mean on $(0,\infty)$, namely if so are M and N.

In a similar way we obtain

COROLLARY 3. *Let $M, N:(0,\infty)^2 \to (0,\infty)$ be arbitrary means on $(0,\infty)$, and $f:(0,\infty) \to \mathbf{R}$ a function. Then $K:(0,\infty)^2 \to (0,\infty)$ defined by*

$$K(x,y) := f(M(x,y) + N(x,y)),$$

is a mean on $(0,\infty)$ if, and only if, $f(x) = x/2$ for all $x > 0$, and, consequently,

$$K(x,y) = \frac{M(x,y) + N(x,y)}{2}, \qquad x,y>0.$$

Suppose that M and N are positively homogeneous and quasi-arithmetic means. Then K is quasi-arithmetic if, and only if, $M = N(= K)$.

Remark 7. Suppose that M and N are arbitrary means on $(0,\infty)$, and $g:(0,\infty) \to (0,\infty)$ a function. If $K:(0,\infty)^2 \to (0,\infty)$,

$$K(x,y) := g(M(x,y)) - N(x,y), \qquad x,y>0,$$

is a mean on $(0,\infty)$, then $g(x) = 2x$, $x > 0$, and consequently,

$$K(x,y) = 2M(x,y) - N(x,y), \qquad x,y>0.$$

Note that Corollary 3 answers the question when the function $2M - N$ is a positively homogeneous and quasi-arithmetic mean on $(0,\infty)$, namely if so are M and N.

REFERENCES

1. J. Aczél "Lectures on Functional Equations and Their Applications," Academic Press, New York/London, 1966.
2. J. Aczél and J. Dhombres, Functional equations in several variables, *in* "Encyclopedia of Mathematics and Its Applications," Vol. 31, Cambridge Univ. Press, Cambridge, UK, 1989.
3. J. Aczél and Gy. Maksa, Problem 1, *Aequationes Math.* **51** (1996), 161–162.