

Convex functions with respect to an arbitrary mean

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Abstract

For a mean M , a notion of M -convex function is introduced. A general criterion for the M -convexity of the sum of M -convex functions is given. As an application, we present conditions under which polynomials and the exponential functions are convex with respect to some of the Stolarsky means.

Introduction

Let $J \subset \mathbb{R}$ be a fixed open interval and $M : J \times J \longrightarrow J$ a mean in J , i.e.,

$$\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}, \quad x, y \in J.$$

Let $I \subset J$ be an arbitrary open interval. A function $\phi : I \longrightarrow J$ is said to be *convex with respect to M on I* (shortly *M -convex on I*) iff

$$\phi(M(x, y)) \leq M(\phi(x), \phi(y)), \quad x, y \in I.$$

Taking in the above definition $J = \mathbb{R}$ and M the arithmetic mean, $M(x, y) = (x + y)/2$, we get the notion of *Jensen convex function*. The theory of Jensen convex functions, strictly related to the classical convexity, is important and well known (cf. for instance M. Kuczma [2], p. 122). This theory can be easily carried out to the M -convex functions if M is a *quasi-arithmetic mean*, i.e., if M is of the form

$$M(x, y) = f^{-1} \left(\frac{f(x) + f(y)}{2} \right), \quad x, y \in J,$$

where $f : J \longrightarrow \mathbb{R}$ is a continuous and strictly monotonic function. Actually, it is easy to check that *if f is increasing (decreasing) then ϕ is M -convex on $I \subset J$ iff the function $f \circ \phi \circ f^{-1}$ is Jensen convex (Jensen concave) on $f(I)$* . According to our best knowledge there is no theory of M -convex functions yet where M is not the (weighted) arithmetic or a quasi-arithmetic mean.

The only exception is [8] where the problem of M -convexity of the family of power functions is considered. In [8] some earlier known inequalities are interpreted as

M -convexity of special power functions with respect to a suitable mean M . This approach permits to obtain natural generalizations as well as to look at some inequalities from a more systematic point of view.

In this paper we give a general criterion for M -convexity of the sum of M -convex functions (Theorem 1). Its version for a positively homogeneous mean has easy to verify assumptions. As an application we give conditions under which special polynomials and the exponential functions are convex with respect to some of the Stolarsky [10] means. The power means and the Gini means are also mentioned in this context.

1. Superadditivity of mean M and M -convexity of sum of M -convex functions

Let $J \subset \mathbb{R}$ be a fixed open interval, $M : J \times J \rightarrow J$ a mean in J , and $I \subset J$ an arbitrary open subinterval. A function $\phi : I \rightarrow J$ is said to be M -convex on I iff

$$\phi(M(x, y)) \leq M(\phi(x), \phi(y)), \quad x, y \in I,$$

and strictly M -convex on I iff

$$\phi(M(x, y)) < M(\phi(x), \phi(y)), \quad x, y \in I, x \neq y.$$

If the inequalities are reversed, the function ϕ is said to be M -concave on I or strictly M -concave on I , respectively. If ϕ is both M -convex and M -concave on I , i.e.,

$$\phi(M(x, y)) = M(\phi(x), \phi(y)), \quad x, y \in I,$$

it is called M -affine on I .

These definitions are correct because $M(I, I) = I$ for all intervals $I \subseteq J$.

Remark 1. Let $M : J \times J \rightarrow J$ be an arbitrary mean. Then the identity function $\phi(x) = x$, $x \in J$ and, for every $c \in J$, the constant function $\phi(x) = c$, $x \in J$, are M -affine on J .

Let us note the following obvious

Remark 2. Let $J \subset \mathbb{R}$ be open an interval, $M : J \times J \rightarrow J$ a mean on J , and suppose that $I \subset J$ is an open subinterval.

1°. If $\phi : I \rightarrow J$ and $\psi : J \rightarrow J$ are M -convex and ψ is increasing, then $\psi \circ \phi$ is M -convex.

2°. Let $\phi : I \rightarrow J$ be bijective and (strictly) M -convex.

If ϕ is increasing, then ϕ^{-1} is (strictly) M -concave on J .

If ϕ is decreasing, then ϕ^{-1} is strictly M -convex on J .

In the sequel we put $\mathbb{R}_+^* := (0, \infty)$.

Definition. A function $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ is said to be *superadditive* on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ iff

$$M(x_1, y_1) + M(x_2, y_2) \leq M(x_1 + x_2, y_1 + y_2), \quad x_1, x_2, y_1, y_2 > 0,$$

and *subadditive* iff the inequality is reversed.

Theorem 1. Let $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ be a superadditive mean and $I \subset \mathbb{R}_+^*$ an open interval. If $\phi, \psi : I \longrightarrow \mathbb{R}_+^*$ are M -convex then $\phi + \psi$ is M -convex; if moreover, ϕ or ψ is strictly M -convex, then so is $\phi + \psi$.

Proof. Applying in turn M -convexity of ϕ and ψ , and superadditivity of M , we obtain

$$\begin{aligned} (\phi + \psi)(M(x, y)) &= \phi(M(x, y)) + \psi(M(x, y)) \\ &\leq M(\phi(x), \phi(y)) + M(\psi(x), \psi(y)) \\ &\leq M(\phi(x) + \psi(x), \phi(y) + \psi(y)) \\ &= M((\phi + \psi)(x), (\phi + \psi)(y)) \end{aligned}$$

for all $x, y \in I$, which means that $\phi + \psi$ is M -convex. In view of the definition of strict convexity, the remaining statement is obvious.

Remark 3. To get the counterpart of this result for M -concave functions it is enough to replace the superadditivity of M by its subadditivity, and M -convexity of ϕ and ψ by M -concavity.

In connection with Theorem 1, let us note the following

Remark 4. For a mean $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ and an open interval $I \subset \mathbb{R}_+^*$, denote by $\text{Conv}_M(I)$ the family of all M -convex functions $\phi : I \longrightarrow \mathbb{R}_+^*$. If the family $\text{Conv}_M(I)$ has the two properties

1) for all $x, y \in \mathbb{R}_+^*$ and for all $s, t \in I$, $s \neq t$, there exists $\phi \in \text{Conv}_M(I)$ such that

$$\phi(s) = x, \quad \phi(t) = y, \quad \phi(M(s, t)) = M(\phi(s), \phi(t));$$

2) for all $\phi, \psi \in \text{Conv}_M(I)$, $\phi + \psi \in \text{Conv}_M(I)$,

then M is superadditive on $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

Proof. Take arbitrary $x_1, x_2, y_1, y_2 > 0$ and $s, t \in I$, $s \neq t$. According to 1) there exist $\phi_1, \phi_2 \in \text{Conv}_M(I)$ such that

$$\phi_i(s) = x_i, \quad \phi_i(t) = y_i, \quad \phi_i(M(s, t)) = M(\phi_i(s), \phi_i(t)) = M(x_i, y_i),$$

for $i = 1, 2$. Hence, making use of the relation $\phi_1 + \phi_2 \in \text{Conv}_M(I)$, we get

$$\begin{aligned} M(x_1, y_1) + M(x_2, y_2) &= \phi_1(M(s, t)) + \phi_2(M(s, t)) \\ &= (\phi_1 + \phi_2)(M(s, t)) \\ &\leq M((\phi_1 + \phi_2)(s), (\phi_1 + \phi_2)(t)) \\ &= M(\phi_1(s) + \phi_2(s), \phi_1(t) + \phi_2(t)) \\ &= M(x_1 + x_2, y_1 + y_2), \end{aligned}$$

which was to be shown.

The superadditivity (resp. subadditivity) of a mean M plays a key role in Theorem 1. Since there is no general criterion of superadditivity (subadditivity) of functions of two variables, it is in many cases a nontrivial question to decide if M has this property. Now we shall show that if M is positively homogeneous, the situation is not difficult.

For a two place function $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ we denote the function $\mathbb{R}_+^* \ni x \rightarrow M(x, 1)$ by $M(\cdot, 1)$. Similarly we define the function $M(1, \cdot)$.

Let us quote the following result (cf. [6], Theorem 9):

Lemma 1. *Let $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a positively homogeneous function, i.e.,*

$$M(tx, ty) = tM(x, y), \quad t, x, y > 0.$$

Then M is superadditive if, and only if, the function $M(\cdot, 1)$ (or $M(1, \cdot)$) is concave.

Proof. Suppose that $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ is positively homogeneous. Putting $h := M(\cdot, 1)$ we can write

$$M(x, y) = y h\left(\frac{x}{y}\right), \quad x, y > 0,$$

Now it is easily seen that h is concave if, and only if (cf. [4]),

$$(y_1 + y_2)h\left(\frac{x_1 + x_2}{y_1 + y_2}\right) \geq y_1 h\left(\frac{x_1}{y_1}\right) + y_2 h\left(\frac{x_2}{y_2}\right), \quad x_1, x_2, y_1, y_2 > 0,$$

i.e., if, and only if, the function M is superadditive.

A more general inequality as well as its integral counterpart are presented in [5], [9], and in [7] where also the equality case is considered.

Now we can prove

Theorem 2. Let $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a positively homogeneous mean. If the function $M(\cdot, 1)$ (or $M(1, \cdot)$) is concave, and $I \subset \mathbb{R}_+^*$ is an open interval, then

- 1) for every two M -convex functions $\phi, \psi : I \rightarrow \mathbb{R}_+^*$, the function $\phi + \psi$ is M -convex, and it is strictly M -convex if ϕ or ψ is strictly M -convex;
- 2) for every (strictly) M -convex $\phi : I \rightarrow \mathbb{R}_+^*$ and $a > 0$, the function $a\phi$ is (strictly) M -convex.

Proof. Part 1) is a consequence of Theorem 1 and Lemma 1. Part 2) follows immediately from the positive homogeneity of M .

Put $\mathbb{N}^0 := \mathbb{N} \cup \{0\}$.

Corollary 1. Let $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a positively homogeneous mean such that the function $M(\cdot, 1)$ (or $M(1, \cdot)$) is concave, and $I \subset \mathbb{R}_+^*$ an open interval. Suppose that for every $k \in \mathbb{N}^0$, $\phi_k : I \rightarrow \mathbb{R}_+^*$ is M -convex on I , and $c_k > 0$. Then for every $n \in \mathbb{N}^0$, the function

$$f_n := \sum_{k=0}^n c_k \phi_k$$

is M -convex on I , and, if there is a $k \in \{0, 1, \dots, n\}$ such that ϕ_k is strictly M -convex, then f_n is strictly M -convex on I . Moreover, if the series

$$f := \sum_{k=0}^{\infty} c_k \phi_k$$

is pointwise convergent on I , then f is M -convex on I , and it is strictly M -convex if at least one of the functions ϕ_k is strictly M -convex.

Proof. By Theorem 2, the function f_n is M -convex on I , i.e.,

$$f_n(M(x, y)) \leq M(f_n(x), f_n(y)), \quad x, y \in I,$$

for every $n \in \mathbb{N}^0$. The function $M(\cdot, 1)$, being concave, is continuous. Since $M(x, y) = yM(x/y, 1)$, $x, y > 0$, the function M is continuous. Hence, letting $n \rightarrow \infty$ in the above inequality, we obtain

$$f(M(x, y)) \leq M(f(x), f(y)), \quad x, y \in I,$$

which completes the proof.

Remark 5. Obviously Theorem 1 and Lemma 1 remain valid on replacing superadditivity of M by subadditivity of M , and M -convexity of functions ϕ and ψ by M -concavity. Therefore the suitable counterparts of Theorem 2 and Corollary 1 also hold true.

2. Convex functions with respect to the Stolarsky means

An important class of means, strictly related to the Cauchy mean-value theorem, is the two parameter family of Stolarsky means $E_{r,s} : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ ($r, s \in \mathbb{R}$) defined by (cf. [10], also [1], p. 345)

$$E_{r,s}(x, y) := \begin{cases} \left(\frac{r}{s} \cdot \frac{x^s - y^s}{x^r - y^r} \right)^{1/(s-r)}, & r \neq s, \quad rs \neq 0; \quad x \neq y \\ \left(\frac{1}{r} \cdot \frac{x^r - y^r}{\log x - \log y} \right)^{\frac{1}{r}}, & r \neq 0; \quad s = 0; \quad x \neq y \\ \left(\frac{1}{s} \cdot \frac{x^s - y^s}{\log x - \log y} \right)^{\frac{1}{s}}, & r = 0; \quad s \neq 0; \quad x \neq y \\ \left(e^{-\frac{1}{r}} \right) \cdot \left(\frac{y^r}{x^r} \right)^{1/(y^r - x^r)}, & r = s \neq 0; \quad x \neq y \\ \sqrt{xy}, & r = s = 0; \quad x \neq y \\ x, & r, s \in \mathbb{R}; \quad x = y \end{cases}$$

The family $(E_{r,s})_{r,s \in \mathbb{R}}$ of positively homogeneous means contains the arithmetic $(E_{2,1})$, geometric $(E_{0,0})$, harmonic $(E_{-2,-1})$, and logarithmic $(E_{1,0})$ means as some special cases.

Recently Losonczi and Páles [3] gave conditions under which the means $E_{r,s}$ are superadditive or subadditive. We quote their result as

Lemma 2. Let $r, s \in \mathbb{R}$, $r \neq s$, be fixed.

1°. $E_{r,s}$ is superadditive on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ if $\min(r, s) \leq 1$ and $r + s \leq 3$.

2°. $E_{r,s}$ is subadditive on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ if $\min(r, s) \geq 1$ and $r + s \geq 3$.

In [8] (Theorem 13) we have proved the following result.

Lemma 3. Let $r, s \in \mathbb{R}$.

1°. If $r + s > 0$, then

$\mathbb{R}_+^* \ni x \longrightarrow x^p$ is strictly $E_{r,s}$ -convex on \mathbb{R}_+^* for $p \in \mathbb{R} \setminus [0, 1]$,

$\mathbb{R}_+^* \ni x \longrightarrow x^p$ is strictly $E_{r,s}$ -concave on \mathbb{R}_+^* for $p \in (0, 1)$.

2°. If $r + s < 0$, then

$\mathbb{R}_+^* \ni x \longrightarrow x^p$ is strictly $E_{r,s}$ -concave on \mathbb{R}_+^* for $p \in \mathbb{R} \setminus [0, 1]$,

$\mathbb{R}_+^* \ni x \longrightarrow x^p$ is strictly $E_{r,s}$ -convex on \mathbb{R}_+^* for $p \in (0, 1)$.

3°. If $r + s = 0$, then $\mathbb{R}_+^* \ni x \longrightarrow x^p$ is $E_{r,s}$ -affine on \mathbb{R}_+^* for all $p \in \mathbb{R}$.

Applying Lemma 1, Lemma 2, Lemma 3 and Corollary 1 gives

Theorem 3. Let $r, s \in \mathbb{R}$, $r \neq s$, such that

$$\min(r, s) \leq 1, \quad r + s \leq 3,$$

be fixed. Suppose that $p_k \in \mathbb{R} \setminus (0, 1)$ and $c_k > 0$, for every $k \in \mathbb{N}^0$. Then for every $n \in \mathbb{N}^0$, the function $f_n : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$,

$$f_n(x) := \sum_{k=0}^n c_k x^{p_k}$$

is $E_{r,s}$ -convex on \mathbb{R}_+^* . Moreover, if the series

$$f(x) := \sum_{k=0}^{\infty} c_k x^{p_k}$$

is pointwise convergent on an interval $I \subset \mathbb{R}_+^*$, then the function f is $E_{r,s}$ -convex on I .

Hence, applying Remark 2.2°, we get

Corollary 2. Let $r, s \in \mathbb{R}$, $r \neq s$, such that

$$\min(r, s) \leq 1, \quad r + s \leq 3,$$

be fixed. Then

- 1°. every polynomial

$$f_n(x) := \sum_{k=0}^n c_k x^k, \quad x \in \mathbb{R}_+^*,$$

such that

$$c_k \geq 0, \quad k = 0, 1, \dots, n; \quad \sum_{k=0}^n c_k > 0,$$

is $E_{r,s}$ -convex on \mathbb{R}_+^* ;

2°. for every $a > 1$ the exponential function $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$,

$$f(x) = a^x, \quad x > 0,$$

is strictly $E_{r,s}$ -convex on \mathbb{R}_+^* ;

3°. for every $a > 1$ the function $f = \log_a$ is strictly $E_{r,s}$ -concave on the interval $(1, \infty)$.

The assumption $a > 1$ is needed for applying Theorem 3 in Corollary 2.2° since

$$a^x = \sum_{k=0}^{\infty} c_k x^k \quad \text{with } c_k = \frac{(\log a)^k}{k!} > 0 \quad \text{for all } k \in \mathbb{N}^0;$$

here \log denotes the natural logarithm. We show now that $a > 1$ is really essential for part 2° of Corollary 2.

Remark 6. Let $a \in (0, 1)$ be fixed. Then the exponential function $f(x) = a^x$, $x > 0$, is neither $E_{1,0}$ -convex nor $E_{1,0}$ -concave on \mathbb{R}_+^* .

Proof. We first show that the function $h(x) := e^{-x}$, $x > 0$, is neither $E_{1,0}$ -convex nor $E_{1,0}$ -concave on \mathbb{R}_+^* , i.e., that neither the inequality

$$h\left(\frac{x-y}{\log(x)-\log(y)}\right) \leq \frac{x-y}{\log h(x)-\log h(y)}, \quad x, y > 0,$$

nor the reverse inequality holds true. For $y := e^2$ put

$$\alpha(x) := h\left(\frac{x-y}{\log(x)-\log(y)}\right), \quad \beta(x) := \frac{h(x)-h(y)}{\log h(x)-\log h(y)}, \quad x > 0, x \neq e^2.$$

Then

$$\alpha(x) = \exp\left(-\frac{x-e^2}{\log(x)-2}\right), \quad \beta(x) = \frac{e^{-x}-e^{-e^2}}{-x+e^2}, \quad x > 0, x \neq e^2.$$

Since

$$\alpha(0+) := \lim_{x \rightarrow 0+} \alpha(x) = 1, \quad \beta(0) = \frac{1-e^{-e^2}}{e^2},$$

we have $\alpha(0+) > \beta(0)$. This shows that h cannot be $E_{1,0}$ -convex on \mathbb{R}_+^* .

On the other hand we have

$$\alpha(1) = e^{\frac{1-e^2}{2}}, \quad \beta(1) = \frac{e^{-1}-e^{-e^2}}{e^2-1}.$$

We shall show that

$$\alpha(1) < \beta(1).$$

It is easy to see that this is equivalent to the following inequality

$$e^{\frac{3-e^2}{2}} - e^{\frac{1-e^2}{2}} < e^{-1} - e^{-e^2}.$$

This inequality holds true because

$$e^{\frac{3-e^2}{2}} < e^{-1}, \quad e^{\frac{1-e^2}{2}} > e^{-e^2}.$$

In fact, the first of these inequalities is equivalent to $e^2 > 7$, and the second one is obvious.

Suppose now that there is an $a \in (0, 1)$ such that the function $f(x) = a^x$, $x > 0$, is $E_{1,0}$ -convex (or $E_{1,0}$ -concave) on \mathbb{R}_+^* . Choose $c > 0$ such that $a^c = e^{-1}$. Then, putting $g(x) := cx$, $x > 0$, we would have $h = f \circ g$. Since g is $E_{1,0}$ -affine on \mathbb{R}_+^* , the function h is $E_{1,0}$ -convex (or $E_{1,0}$ -concave) on \mathbb{R}_+^* by Remark 2.1⁰. This contradiction completes the proof.

Corollary 2 allows to produce a lot of new inequalities. Consider for instance the following

Example. Taking $r = 1$, $s = 0$, and $f(x) = e^x$, $x > 0$, in Corollary 2.2^o gives

$$f(E_{1,0}(x, y)) \leq E_{1,0}(f(x), f(y)), \quad x, y > 0.$$

By the definition of the logarithmic mean $E_{1,0}$ we hence get the inequality

$$e^{\frac{x-y}{\log(x)-\log(y)}} \leq \frac{e^x - e^y}{x - y}, \quad x, y > 0, x \neq y.$$

3. Remarks about some other means

Let us fix $p \in \mathbb{R}$. The function $A_p : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ defined by

$$A_p(x, y) := \begin{cases} \left(\frac{x^p + y^p}{2} \right)^{\frac{1}{p}}, & p \neq 0 \\ \sqrt{xy}, & p = 0 \end{cases}.$$

is a symmetric positively homogeneous mean, and it is called the *power mean*. Moreover, $A_0(\cdot, 1)$ is concave, and for $p \neq 0$

$$\frac{d^2}{dx^2} A_p(x, 1) = x^{p-2} (x^p + 1)^{-2} A_p(x, 1) (p-1), \quad x > 0,$$

so the function $A_p(\cdot, 1)$ is concave for every $p \leq 1$ and convex for every $p \geq 1$.

Applying Theorem 1 and Corollary 1, it is easy to get the counterparts of Theorem 3 and Corollary 2 for the A_p -convex functions.

Let us also mention that, using some results of Losonczi and Páles [3], one can get more general results about $M_{r,s}$ -convex functions, where $M_{r,s} : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$, is the two parameter family of Gini means, which for $r \neq s$ is defined by

$$M_{r,s}(x, y) := \left(\frac{x^r + y^r}{x^s + y^s} \right)^{1/(r-s)}, \quad x, y > 0.$$

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