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# Convexity of power functions with respect to symmetric homogeneous means

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**Abstract**

It is well known that the power function $x \mapsto x^p$ on $\mathbb{R}_+^*$ is strictly Jensen-convex if $p^2 - p > 0$, strictly Jensen-concave if $p^2 - p < 0$, Jensen-convex and Jensen-concave if $p^2 - p = 0$. These Jensen type properties are based upon the arithmetic mean $A : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$. It is the purpose of this paper to investigate the convexity/concavity classification of the power functions for symmetric homogeneous means on $\mathbb{R}_+^*$ other than $A$. In Section 3, a convexity/concavity criterion is presented, and in Section 4 this is applied to the families of Stolarsky means and Gini means (both containing $A$) as well as to weighted geometric means.

## 1. Introduction

Throughout the paper, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$, $\mathbb{R}_+^*$, $\mathbb{R}_-^*$, $\mathbb{R}^*$ denote the sets of real, nonnegative real, nonpositive real, positive real, negative real, non-zero real numbers, respectively. For $p \in \mathbb{R}$ we define

$$\varphi_p : \mathbb{R}_+^* \to \mathbb{R}_+^*, \qquad \varphi_p(x) := x^p \qquad (x \in \mathbb{R}_+^*). \tag{1}$$

*log* denotes the natural logarithm.

**Definition 1.**

a) The function $M : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ is called a *mean* on $\mathbb{R}_+^*$ if

$$\begin{aligned} &\min\{x,y\} \le M(x,y) \le \max\{x,y\}, \quad \text{i.e.,} \\ &M(x,y) \in \operatorname{conv}\{x,y\}, \quad \text{for all } x,y \in \mathbb{R}_+^*. \end{aligned} \tag{2}$$

b) A mean $M$ on $\mathbb{R}_+^*$ is said to be *homogeneous* (more precisely: *positively homogeneous*) if

$$M(ax, ay) = aM(x, y) \qquad \text{for all } a, x, y \in \mathbb{R}_+^*, \tag{3}$$

*symmetric* if

$$M(x, y) = M(y, x) \text{ for all } x, y \in \mathbb{R}_+^*. \tag{4}$$

The set of all symmetric homogeneous means on $\mathbb{R}_+^*$ is denoted by $\mathcal{M}(\mathbb{R}_+^*)$. Note that the degree of homogeneity of a mean has to be one.

**Definition 2.** If $M$ is a mean on $\mathbb{R}_+^*$, a function $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is called

a) *$M$-convex* on $\mathbb{R}_+^*$ if

$$\varphi(M(x, y)) \le M(\varphi(x), \varphi(y)) \text{ for all } x, y \in \mathbb{R}_+^*, \tag{5}$$

b) *strictly $M$-convex* on $\mathbb{R}_+^*$ if

$$\varphi(M(x, y)) < M(\varphi(x), \varphi(y)) \text{ for all } x, y \in \mathbb{R}_+^*,\ x \ne y, \tag{6}$$

c) *$M$-concave* on $\mathbb{R}_+^*$ if

$$\varphi(M(x, y)) \ge M(\varphi(x), \varphi(y)) \text{ for all } x, y \in \mathbb{R}_+^*, \tag{7}$$

d) *strictly $M$-concave* on $\mathbb{R}_+^*$ if

$$\varphi(M(x, y)) > M(\varphi(x), \varphi(y)) \text{ for all } x, y \in \mathbb{R}_+^*,\ x \ne y, \tag{8}$$

e) *$M$-affine* on $\mathbb{R}_+^*$ if

$$\varphi(M(x, y)) = M(\varphi(x), \varphi(y)) \text{ for all } x, y \in \mathbb{R}_+^*. \tag{9}$$

**Remark 3.** It is clear that the convexity/concavity behavior of $\varphi_p$ is strongly related to the comparison (also called dominance) problem of the means involved: strict $M$-convexity of $\varphi_p$ for $p \in \mathbb{R}^*$ on $\mathbb{R}_+^*$, e.g., means that

$$M(x, y) < (M(x^p, y^p))^{\frac{1}{p}} \qquad (x, y \in \mathbb{R}_+^*;\ x \ne y)$$

which expresses a comparison result for two means being "conjugate by $\varphi_p$". So our convexity/concavity problem is a special case of the general comparison problem. The proofs here are easier and more direct for our purposes and, accordingly, the structure of the conditions is simpler that those for the general problem (cf., e.g., [9] , [14], [13] in connection with some cases in Theorems 13 and 16). The main reason for our direct access, however, is that we intend to obtain strict inequalities also in the limiting cases (e.g., for $E_{r,r}$ and $E_{r,0}$ in Theorem 13), and this does require an extra effort anyway.

**Remark 4.** For a different point of view leading to the functional inequalities in Definition 2, cf., e.g. [16] , p. 38–39 or [18], p. 103–104.

## 2. Preliminaries

**Lemma 5.** *Let $M$ be a mean on $\mathbb{R}_+^*$.*

1°. *If $\varphi: \mathbb{R}_+^* \to \mathbb{R}_+^*$ and $\psi: \mathbb{R}_+^* \to \mathbb{R}_+^*$ are*

$$M\text{-}\substack{\text{convex}\\ \text{concave}} \quad \text{on } \mathbb{R}_+^*$$

*and $\psi$ is increasing, then $\psi \circ \varphi$ is $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$. (There is no analogue for $\psi$ decreasing.)*

2°. *Let $\varphi: \mathbb{R}_+^* \to \mathbb{R}_+^*$ be bijective and (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$.*
*If $\varphi$ is increasing, then $\varphi^{-1}$ is (strictly) $M\text{-}\substack{\text{concave}\\ \text{convex}}$ on $\mathbb{R}_+^*$.*
*If $\varphi$ is decreasing, then $\varphi^{-1}$ is (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$.*

The immediate proof is omitted.

The next statement prepares the convexity/concavity classification of the power functions $\varphi_p$ ($p \in \mathbb{R}$) and focusses the interest to $p > 1$ and $p = -1$.

**Theorem 6.** *Let $M$ be a mean on $\mathbb{R}_+^*$.*

1°. *For every $p > 1$, the function $\varphi_p$ is (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$ if, and only if, the function $\varphi_{\frac{1}{p}}$ is (strictly) $M\text{-}\substack{\text{concave}\\ \text{convex}}$ on $\mathbb{R}_+^*$.*

2°. *If all $\varphi_p$ ($p > 1$) are (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$ and if $\varphi_{-1}$ is $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$, then all $\varphi_q$ ($q < 0$) are (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$.*

3°. *If all $\varphi_p$ ($p > 1$) are $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$ and if $\varphi_{-1}$ is (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$, then all $\varphi_q$ ($q < 0$) are (strictly) $M\text{-}\substack{\text{convex}\\ \text{concave}}$ on $\mathbb{R}_+^*$.*

4°. *$\varphi_0$ and $\varphi_1$ are $M$-affine on $\mathbb{R}_+^*$.*

*Proof.* 1°. is obtained by the substitution $u = x^p, v = y^p$.

2°. Assume that all $\varphi_p$ ($p > 1$) are strictly $M$-convex and $\varphi_{-1}$ is $M$-convex on $\mathbb{R}_+^*$. Suppose first $q < -1$. Then $-q > 1$, and by the assumption we have

$$(M(x,y))^{-q} < M(x^{-q}, y^{-q}) \qquad (x, y \in \mathbb{R}_+^*, x \neq y).$$

Substituting $x$, $y$ by $x^{-1}$, $y^{-1}$ we get

$$(M(x^{-1}, y^{-1}))^{-q} < M(x^q, y^q) \qquad (x, y \in \mathbb{R}_+^*,\ x \neq y). \tag{10}$$

$M$-convexity of $\varphi_{-1}$ on $\mathbb{R}_+^*$ yields

$$(M(x,y))^{-1} \leq M(x^{-1}, y^{-1}) \qquad (\forall\, x, y \in \mathbb{R}_+^*,\ x \neq y).$$

Since $\varphi_{-q}$ is strictly increasing, it follows that

$$(M(x,y))^q \le (M(x^{-1},y^{-1}))^{-q} \qquad (\forall\, x,y \in \mathbb{R}_+^*,\ x \ne y)$$

and together with (10)

$$(M(x,y))^q < M(x^q,y^q) \qquad (\forall\, x,y \in \mathbb{R}_+^*,\ x \ne y), \tag{11}$$

i.e., that $\varphi_q$ is strictly $M$-convex on $\mathbb{R}_+^*$.

Now suppose that $q \in ]-1,0[$. Then $-\frac{1}{q} > 1$, and by the assumption, $\varphi_{-\frac{1}{q}}$ is strictly $M$-convex on $\mathbb{R}_+^*$, and since it is increasing, $\varphi_{-q}$ is strictly $M$-concave on $\mathbb{R}_+^*$ by Lemma 5.2°, i.e.,

$$(M(x,y))^{-q} > M(x^{-q},y^{-q}) \qquad (x,y \in \mathbb{R}_+^*,\ x \ne y). \tag{12}$$

$M$-convexity of $\varphi_{-1}$ on $\mathbb{R}_+^*$ implies

$$(M(x^{-q},y^{-q}))^{-1} \le M(x^q,y^q) \qquad (x,y \in \mathbb{R}_+^*,\ x \ne y),$$

and together with (12) we obtain again (11).

The non-strict convexity case as well as the two concavity cases can be proved along the same lines.

3°. The proof follows again the same lines, but here the possible strictness of $\varphi_q$ results from that of $\varphi_{-1}$.

4°. is obvious, and the proof of Theorem 6 is complete. □

**Lemma 7.** *If $M$ is a mean on $\mathbb{R}_+^*$, $p \in \mathbb{R}$, $r \in \mathbb{R}^*$, and ${}_rM(x,y) := (M(x^r,y^r))^{\frac{1}{r}}$ for all $x,y \in \mathbb{R}_+^*$, then we have:*

1°. *${}_rM$ is a mean on $\mathbb{R}_+^*$, the* $r$-conjugate *of $M$.*

2°. *If $M \in \mathcal{M}(\mathbb{R}_+^*)$, then ${}_rM \in \mathcal{M}(\mathbb{R}_+^*)$.*

3°. *If $r \in \mathbb{R}^*$, then $\varphi_p$ is ${}_rM$-affine on $\mathbb{R}_+^* \iff \varphi_p$ is $M$-affine on $\mathbb{R}_+^*$.*

4°. *If $r \in \mathbb{R}_+^*$, then $\varphi_p$ is ${}_rM$-$\substack{convex\\concave}$ on $\mathbb{R}_+^* \iff \varphi_p$ is $M$-$\substack{convex\\concave}$ on $\mathbb{R}_+^*$.*

5°. *If $r \in \mathbb{R}_-^*$, then $\varphi_p$ is ${}_rM$-$\substack{convex\\concave}$ on $\mathbb{R}_+^* \iff \varphi_p$ is $M$-$\substack{concave\\convex}$ on $\mathbb{R}_+^*$.*

The obvious proof is omitted.

### 3. An M-convexity/-concavity criterion

After having dealt with arbitrary means on $\mathbb{R}_+^*$ in Section 2, we now turn to the question as to how we can recognize whether all power functions $\varphi_p (p > 1)$ have the same convexity/concavity behavior with respect to a symmetric homogeneous mean on $\mathbb{R}_+^*$.

**Theorem 8.** *If $M \in \mathcal{M}(\mathbb{R}_+^*)$, then all $\varphi_p$ $(p > 1)$ are (strictly) $M$- $\genfrac{}{}{0pt}{}{convex}{concave}$ on $\mathbb{R}_+^*$ if, and only if the function*

$$f_M : \mathbb{R}_+^* \to \mathbb{R}_+^*, \qquad f_M(x) := (M(e^x, 1))^{\frac{1}{x}} \qquad (x \in \mathbb{R}_+^*) \tag{13}$$

*is (strictly) $\genfrac{}{}{0pt}{}{increasing}{decreasing}$ on $\mathbb{R}_+^*$.*

*Proof.* We write the argument for the strictly convex case; immediate modifications for the three remaining cases are available.

Using symmetry of $M$, putting $t = \frac{x}{y}$, using homogeneity of $M$, dividing by $y^p$, putting $e^x = t$, and exponentiating with $\frac{1}{px}$ $(> 0)$ generate the following sequence of mutually equivalent statements:

$$\begin{array}{lcll}
(M(x,y))^p & < & M(x^p, y^p) & (x, y \in \mathbb{R}_+^*;\ x \neq y; p > 1) \\
(M(x,y))^p & < & M(x^p, y^p) & (x, y \in \mathbb{R}_+^*;\ x > y; p > 1) \\
(M(ty,y))^p & < & M(t^p y^p, y^p) & (t > 1;\ y \in \mathbb{R}_+^*;\ p > 1) \\
y^p (M(t,1))^p & < & y^p M(t^p, 1) & (t > 1;\ y \in \mathbb{R}_+^*;\ p > 1) \\
(M(t,1))^p & < & M(t^p, 1) & (t > 1,\ p > 1) \\
(M(e^x,1))^p & < & M(e^{px}, 1) & (x \in \mathbb{R}_+^*,\ p > 1) \\
(M(e^x,1))^{\frac{1}{x}} & < & (M(e^{px}, 1))^{\frac{1}{px}} & (x \in \mathbb{R}_+^*,\ p > 1) \\
f_M(x) & < & f_M(px) & (x \in \mathbb{R}_+^*,\ p > 1).
\end{array}$$

□

### 4. Applications

We first deal with the quasi-arithmetic means in $\mathcal{M}(\mathbb{R}_+^*)$, which are precisely the geometric mean $G =: M_0$ and the *power means* $M_r$ $(r \in \mathbb{R}^*)$ given by

$$M_r(x,y) := \left(\frac{x^r + y^r}{2}\right)^{\frac{1}{r}} \quad \text{for all } x, y \in \mathbb{R}_+^*$$

([6], p. 68, Theorem 84), the latter being the $r$-conjugates ${}_rA$ of the arithmetic mean $A$ (cf. Lemma 7.1°). The result stated at the beginning of the Abstract, Lemma 7, and the fact that all $\varphi_p$ $(p \in \mathbb{R})$ are $M_0$-affine on $\mathbb{R}_+^*$ lead to

$$p, r \in \mathbb{R} \Rightarrow \left[\varphi_p \text{ is } \begin{array}{c} \text{strictly } M_r\text{-convex} \\ M_r\text{-affine} \\ \text{strictly } M_r\text{-concave} \end{array} \text{ on } \mathbb{R}_+^* \Leftrightarrow r(p^2 - p) \begin{array}{c} > \\ = \\ < \end{array} 0\right]. \tag{14}$$

Later we shall treat two distinct 2-parameter families $(E_{r,s})_{r,s\in\mathbb{R}}$ and $(M_{r,s})_{r,s\in\mathbb{R}}$ of elements of $\mathcal{M}(\mathbb{R}_+^*)$ which contain the means $M_r$ $(r \in \mathbb{R})$; see Remark 17.

We begin by two specific members of the former family, the *logarithmic mean* $L$ and the *identric mean* $I$; $L$ has numerous applications in physics (cf., e.g., [3] p. 376-377; [7]; [12], sect. 8.7, 8.9; [15], sect. 6.2), and it is not a quasi-arithmetic mean ([10], Theorems 2 and 4; [7], Theorem 1).

**Theorem 9.** *Let* $L : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$, *be defined by*

$$L(x,x) := x,\ L(x,y) := \frac{x-y}{\log x - \log y} \qquad (x, y \in \mathbb{R}_+^*,\ x \neq y).$$

*Then we have*

$1°.\quad L \in \mathcal{M}(\mathbb{R}_+^*)$,

$$2°.\quad p \in \mathbb{R} \Rightarrow \left[\varphi_p \text{ is} \begin{array}{c} \textit{strictly } L\textit{-convex} \\ L\textit{-affine} \\ \textit{strictly } L\textit{-concave} \end{array} \textit{on } \mathbb{R}_+^* \iff p^2 - p \begin{array}{c} > \\ = \\ < \end{array} 0\right].$$

*Proof.* 1°. is well known; the intermediacy property (2) easily follows from the Mean Value Theorem.

2°. (i) For $t > 1$ we have $z := \frac{1}{2}\log t > 0$, so, by looking at the Maclaurin series, $z < \sinh z$, i.e., $2z < e^z - e^{-z}$, therefore $\log t < t^{\frac{1}{2}} - t^{-\frac{1}{2}} (t > 1)$ (cf. also [11], p.272, 3.6.15, where a different argument is used). If follows that

$$(\log t)^2 < t - 2 + \frac{1}{t} \qquad (t > 1). \tag{15}$$

Take $x, y \in \mathbb{R}_+^*$, $x > y$ and put $t = \frac{x}{y}$. Then (15) yields $(\log x - \log y)^2 < \frac{x}{y} - 2 + \frac{y}{x} = (x-y)(\frac{1}{y} - \frac{1}{x})$, which can be written in the form $(L(x,y))^{-1} < L(\frac{1}{x}, \frac{1}{y})$ $(x > y > 0)$, and symmetry of $L$ implies that

$$\varphi_{-1} \text{ is strictly } L\text{-convex on } \mathbb{R}_+^*. \tag{16}$$

(ii) $f_L(x) \overset{(13)}{=} (L(e^x, 1))^{\frac{1}{x}} = (\frac{e^x - 1}{x})^{\frac{1}{x}}$ $(x \in \mathbb{R}_+^*)$. We show that

$$f_L \text{ is strictly increasing on } \mathbb{R}_+^*. \tag{17}$$

To this end we define $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_1(x) := \frac{xe^x}{e^x - 1} - 1,\ f_2(x) := \log\frac{e^x - 1}{x}\ (x \in \mathbb{R}_+^*),\ f_1(0) := 0,\ f_2(0) := 0. \tag{18}$$

$f_1, f_2$ are continuous on $\mathbb{R}_+$ and differentiable on $\mathbb{R}_+^*$. From $\frac{e^x-1}{x} > e^{\frac{x}{2}}$ $(x \in \mathbb{R}_+^*)$ we obtain $\frac{1}{x} > \frac{xe^x}{(e^x-1)^2}$ $(x \in \mathbb{R}_+^*)$, therefore

$$f_1'(x) = \frac{e^x(e^x-1) - xe^x}{(e^x-1)^2} > \frac{e^x}{e^x-1} - \frac{1}{x} = f_2'(x) \quad (x \in \mathbb{R}_+^*). \tag{19}$$

Applying the Mean Value Theorem to $f_1 - f_2$ and using (18) and (19) yields

$$f_1(x) > f_2(x) \qquad (x \in \mathbb{R}_+^*). \tag{20}$$

Finally $f_L'(x) = f_L(x) \cdot \frac{1}{x^2} \cdot (f_1(x) - f_2(x)) > 0$ $(x \in \mathbb{R}_+^*)$, by (20), and (17) is proved.

(iii) By (17) and Theorem 8, all $\varphi_p (p > 1)$ are strictly $L$-convex on $\mathbb{R}_+^*$. By Theorem 6.1°, all $\varphi_p$ $(0 < p < 1)$ are strictly $L$-concave on $\mathbb{R}_+^*$, and by (16) and Theorem 6.2°, all $\varphi_p$ $(p < 0)$ are strictly $L$-convex on $\mathbb{R}_+^*$. Theorem 6.4° completes the "$\Longleftarrow$" part of 2°.

(iv) It is sufficient to prove the "$\Longleftarrow$" statements because on both sides of "$\Longleftrightarrow$" we have threefold alternatives, the one on the right-hand side being complete, i.e., a trichotomy. □

**Theorem 10.** *Let* $I : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$, *be defined by*

$$I(x,x) := x, \; I(x,y) := \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{(y-x)}} \qquad (x, y \in \mathbb{R}_+^*, \; x \neq y).$$

*Then we have*

1°. $I \in \mathcal{M}(\mathbb{R}_+^*)$,

$$2^\circ.\; p \in \mathbb{R} \Rightarrow \left[\varphi_p \textit{ is } \begin{array}{c} \textit{strictly } I\textit{-convex} \\ I\textit{-affine} \\ \textit{strictly } I\textit{-concave} \end{array} \textit{ on } \mathbb{R}_+^* \Longleftrightarrow p^2 - p \begin{array}{c} > \\ = \\ < \end{array} 0\right].$$

*Proof.* 1° is well known.

2°. (i) We start from $\frac{2t}{2+t} < \log(1+t)$ $(t \in \mathbb{R}_+^*)$ ([11], p. 273, 3.6.18) and get $2 < (1 + \frac{2}{t})\log(1+t)$ $(t \in \mathbb{R}_+^*)$. For $x, y \in \mathbb{R}_+^*$, $x < y$, $t := \frac{y-x}{x}$, we obtain by a simple calculation $0 < \log I(x,y) + \log I(x^{-1}, y^{-1})$, i.e., $1 < I(x,y) \cdot I(x^{-1}, y^{-1})$ $(x, y \in \mathbb{R}_+^*, x < y)$. By symmetry of $I$, this is valid for all $x \neq y$, so

$$\varphi_{-1} \text{ is strictly } I\text{-convex on } \mathbb{R}_+^*. \tag{21}$$

(ii) $f_I(x) \overset{(13)}{=} (I(e^x, 1))^{\frac{1}{x}} = e^{\frac{e^x}{(e^x-1)} - \frac{1}{x}}$ $(x \in \mathbb{R}_+^*)$.

For $x \in \mathbb{R}_+^*$ we have $\cosh x > 1 + \frac{x^2}{2}$, $e^x + e^{-x} > 2 + x^2$, $e^x - 2 + e^{-x} > x^2$, $\frac{(e^x-1)^2}{e^x} > x^2$, $0 < -\frac{e^x}{(e^x-1)^2} + \frac{1}{x^2}$ and finally $\left(\frac{e^x}{e^x-1} - \frac{1}{x}\right)' > 0$, which guarantees that

$$f_I \text{ is strictly increasing on } \mathbb{R}_+^*. \tag{22}$$

(iii) The proof is completed as for Theorem 9 with (21), (22) instead of (16), (17). □

**Definition 11.** The *Stolarsky means* ([17], p. 88)

$$E_{r,s} : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^* \qquad (r, s \in \mathbb{R})$$

are defined by

$$E_{r,s}(x,x) := x \qquad (x \in \mathbb{R}_+^*)$$

and for $x, y \in \mathbb{R}_+^*$, $x \neq y$, by

$$E_{r,s}(x,y) := \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{\frac{1}{s-r}} \quad \text{if} \quad rs(s-r) \neq 0 \tag{23}$$

$$E_{r,0}(x,y) := E_{0,r}(x,y) := \left(\frac{1}{r} \cdot \frac{y^r - x^r}{\log y - \log x}\right)^{\frac{1}{r}} \quad \text{if} \quad r \neq 0 \tag{24}$$

$$E_{r,r}(x,y) := e^{-\frac{1}{r}} \cdot \left(\frac{y^{y^r}}{x^{x^r}}\right)^{\frac{1}{y^r - x^r}} \quad \text{if} \quad r = s \neq 0 \tag{25}$$

$$E_{0,0}(x,y) := G(x,y). \tag{26}$$

**Remark 12.** a) All $E_{r,s}$ $(r, s \in \mathbb{R})$ are means on $\mathbb{R}_+^*$: they are so-called mean-value means, arising from the Cauchy Mean Value Theorem ([1], p. 345); furthermore $E_{r,s} \in \mathcal{M}(\mathbb{R}_+^*)$.

b) For numerous further properties of $E_{r,s}$ cf.[8], p. 86. Some of them will be needed here, namely

$$E_{r,2r} = M_r \qquad \text{for all } r \in \mathbb{R}^*, \tag{27}$$

$$E_{r,-r} = G \qquad \text{for all } r \in \mathbb{R}, \tag{28}$$

$$E_{r,kr} = {}_rE_{1,k} \qquad \text{for all } r \in \mathbb{R}^*,\ k \in \mathbb{R}, \tag{29}$$

$$E_{1,0} = L, \qquad \text{the logarithmic mean}, \tag{30}$$

$$E_{1,1} = I, \qquad \text{the identric mean}. \tag{31}$$

**Theorem 13.** *For $p, r, s \in \mathbb{R}$ we have*

$$\varphi_p \ \textit{is} \ \begin{matrix} \textit{strictly } E_{r,s}\textit{-convex} \\ E_{r,s}\textit{-affine} \\ \textit{strictly } E_{r,s}\textit{-concave} \end{matrix} \ \textit{on } \mathbb{R}_+^* \iff (r+s)(p^2-p) \begin{matrix} > \\ = \\ < \end{matrix} 0.$$

*Proof.* It is sufficient to prove the "$\Longleftarrow$" statements (see (iv)) in the proof of Theorem 9.

The case $(r+s)(p^2-p)=0$ is settled by Theorem 6.4° ($p^2-p=0$) and (28) together with the remark before (14) ($r+s=0$).

According to the structure of Definition 11 and taking into account that $r=-s$ is already settled, we treat the following three cases separately:

Case 1: $r,s\in\mathbb{R}^*,\ r\neq s, r\neq -s$ (cf.(23)),
Case 2: $r\in\mathbb{R}^*,\ s=0$ (cf.(24)),
Case 3: $r=s\in\mathbb{R}^*$ (cf.(25)).

*Case* 1. (i) By (23) and (13)

$$f_{E_{r,s}}(t)=\left(\frac{r}{s}\cdot\frac{e^{ts}-1}{e^{tr}-1}\right)^{\frac{1}{t(s-r)}}\qquad (t\in\mathbb{R}_+^*).$$

Therefore

$$(\log f_{E_{r,s}}(t))'=\frac{1}{(s-r)t^2}\cdot(g_1(t)-g_2(t))\qquad (t\in\mathbb{R}_+^*),\tag{32}$$

where $g_1,g_2:\mathbb{R}_+\to\mathbb{R}$, $g_1(t):=\frac{tse^{ts}}{e^{ts}-1}-\frac{tre^{tr}}{e^{tr}-1}$ $(t\in\mathbb{R}_+^*)$, $g_1(0):=0$, $g_2(t):=\log\left(\frac{r}{s}\cdot\frac{e^{ts}-1}{e^{tr}-1}\right)$ $(t\in\mathbb{R}_+^*)$, $g_2(0):=0$.

$g_1$ and $g_2$ are continuous on $\mathbb{R}_+$ and differentiable on $\mathbb{R}_+^*$. Furthermore

$$(g_1(t)-g_2(t))'=t\cdot g_2''(t)\quad (t\in\mathbb{R}_+^*),\tag{33}$$

$$g_2''(t)=r^2\cdot\frac{e^{tr}}{(e^{tr}-1)^2}-s^2\cdot\frac{e^{ts}}{(e^{ts}-1)^2}\quad (t\in\mathbb{R}_+^*).\tag{34}$$

Because of (32), (33), $\operatorname{sgn} g_2''(t)$ is of high importance. We define

$$h_t:\mathbb{R}^*\to\mathbb{R},\quad h_t(v):=\frac{v^2e^{tv}}{(e^{tv}-1)^2}\quad (v\in\mathbb{R}^*;\ t\in\mathbb{R}_+^*\ \text{fixed})$$

and get $t^2\cdot h_t(v)=\left(\frac{\frac{t}{2}v}{\sinh\frac{t}{2}v}\right)^2$ $(v\in\mathbb{R}^*; t\in\mathbb{R}_+^*)$.

The Maclaurin series (or strict convexity on $\mathbb{R}_+^*$) of *sinh* shows that $u\mapsto\frac{\sinh u}{u}$ is strictly increasing on $\mathbb{R}_+^*$, so for all $t\in\mathbb{R}_+^*$, the function $h_t$ is strictly decreasing on $\mathbb{R}_+^*$, moreover it is even.

Now by (34),(33)

$$0\ \begin{matrix}<\\>\end{matrix}\ g_2''(t)\qquad (t\in\mathbb{R}_+^*)\qquad \begin{pmatrix}0<|r|<|s|\\0<|s|<|r|\end{pmatrix},$$

$$0\ \begin{matrix}<\\>\end{matrix}\ (g_1(t)-g_2(t))'\qquad (t\in\mathbb{R}_+^*)\qquad \begin{pmatrix}0<|r|<|s|\\0<|s|<|r|\end{pmatrix}.$$

The Mean Value Theorem and $g_1(0) = 0 = g_2(0)$ imply

$$\begin{array}{lll} 0 & \begin{smallmatrix}<\\>\end{smallmatrix} & g_1(t) - g_2(t) \qquad (t \in \mathbb{R}_+^*) \qquad \begin{pmatrix} 0 < |r| < |s| \\ 0 < |s| < |r| \end{pmatrix}, \qquad \text{i.e.,} \\ 0 & < & \dfrac{1}{s^2 - r^2}(g_1(t) - g_2(t)) \qquad (t \in \mathbb{R}_+^*), \qquad \text{i.e., by (32)} \\ 0 & < & \dfrac{1}{r+s}(\log f_{E_{r,s}}(t))' \qquad (t \in \mathbb{R}_+^*), \end{array}$$

therefore by Theorem 8

$$\text{all } \varphi_p (p > 1) \text{ are strictly } \begin{matrix} E_{r,s}\text{-convex} \\ E_{r,s}\text{-concave} \end{matrix} \text{ on } \mathbb{R}_+^* \quad (r + s \begin{smallmatrix}>\\<\end{smallmatrix} 0). \tag{35}$$

(ii) For every $t \in \mathbb{R}_+^*$ we define

$$\tilde{h}_t(v) := \left( \frac{\sinh \frac{t}{2} v}{\frac{t}{2} \cdot v} \right)^2 \qquad (v \in \mathbb{R}^*).$$

Then $\tilde{h}_t$ is strictly increasing on $\mathbb{R}_+^*$ and even, so

$$\tilde{h}_t(s) \begin{smallmatrix}>\\<\end{smallmatrix} \tilde{h}_t(r) \qquad (t \in \mathbb{R}_+^*) \qquad \begin{pmatrix} 0 < |r| < |s| \\ 0 < |s| < |r| \end{pmatrix}.$$

By putting $z := e^t$ we obtain

$$r^2(2 - z^s - z^{-s}) \begin{smallmatrix}<\\>\end{smallmatrix} s^2(2 - z^r - z^{-r}) \qquad (z > 1) \qquad \begin{pmatrix} 0 < |r| < |s| \\ 0 < |s| < |r| \end{pmatrix}. \tag{36}$$

Let be $x, y \in \mathbb{R}_+^*$ arbitrary, $x \neq y$. Then $z := \max\{\frac{x}{y}, \frac{y}{x}\} > 1$, and (36) yields

$$r^2(y^{-s} - x^{-s})(y^s - x^s) \begin{smallmatrix}<\\>\end{smallmatrix} s^2(y^{-r} - x^{-r})(y^r - x^r) \qquad \begin{pmatrix} 0 < |r| < |s| \\ 0 < |s| < |r| \end{pmatrix}.$$

Dividing by the (negative) right-hand side gives

$$\frac{r}{s} \cdot \frac{y^{-s} - x^{-s}}{y^{-r} - x^{-r}} \cdot \frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \begin{smallmatrix}>\\<\end{smallmatrix} 1 \qquad \begin{pmatrix} 0 < |r| < |s| \\ 0 < |s| < |r| \end{pmatrix}.$$

Raising to the power with exponent $\frac{1}{s-r} \begin{smallmatrix}>\\<\end{smallmatrix} 0 \begin{pmatrix} r < s \\ s < r \end{pmatrix}$ and using (23) leads to

$$E_{r,s}(x^{-1}, y^{-1}) \cdot E_{r,s}(x, y) \begin{smallmatrix}>\\<\end{smallmatrix} 1 \qquad \begin{pmatrix} r + s > 0 \\ r + s < 0 \end{pmatrix}, \qquad \text{i.e.,}$$

$$\varphi_{-1} \text{ is strictly } \begin{matrix} E_{r,s}\text{-convex} \\ E_{r,s}\text{-concave} \end{matrix} \text{ on } \mathbb{R}_+^* \qquad (r + s \begin{smallmatrix}>\\<\end{smallmatrix} 0). \tag{37}$$

(iii) In Case 1, the assertion follows from (35), (37), Theorem 6.1° and 2°.

*Case* 2. Here $r+s=r$. $E_{r,0} \overset{(29)}{=} {}_rE_{1,0} \overset{(30)}{=} {}_rL$. By Theorem 9.2° and Lemma 7.3°, 4°, 5°, the assertion follows.

*Case* 3. Here $r+s=2r$. $E_{r,r} \overset{(29)}{=} {}_rE_{1,1} \overset{(31)}{=} {}_rI$. By Theorem 10.2° and Lemma 7.3°, 4°, 5°, the assertion follows. □

**Definition 14.** The *Gini means* $M_{r,s}: \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ $(r,s \in \mathbb{R})$ are defined by

$$M_{r,s}(x,x) := x \qquad (x \in \mathbb{R}_+^*)$$

and for $x, y \in \mathbb{R}_+^*$, $x \neq y$, by

$$M_{r,s}(x,y) = \begin{cases} \left(\frac{x^s+y^s}{x^r+y^r}\right)^{\frac{1}{(s-r)}} & \text{if } r \neq s \\ e^{\frac{x^r \log x + y^r \log y}{x^r+y^r}} & \text{if } r = s \end{cases}$$

([4]; J. Aczél informed the authors about the earlier reference [5]). There are applications in statistics. For further references cf. [2].

**Remark 15.** It is well-known that $M_{r,s} \in \mathcal{M}(\mathbb{R}_+^*)$ for all $r, s \in \mathbb{R}$. Furthermore

$$M_{r,0} = M_r \qquad \text{for all } \ r \in \mathbb{R}^*, \tag{38}$$

$$M_{r,-r} = G \qquad \text{for all } \ r \in \mathbb{R}. \tag{39}$$

**Theorem 16.** *For $p, r, s \in \mathbb{R}$ we have*

$$\varphi_p \text{ is } \begin{array}{c} \textit{strictly } M_{r,s}\textit{-convex} \\ M_{r,s}\textit{-affine} \\ \textit{strictly } M_{r,s}\textit{-concave} \end{array} \quad \text{on } \mathbb{R}_+^* \iff (r+s)(p^2-p) \begin{array}{c} > \\ = \\ < \end{array} 0.$$

*Proof.* As for Theorem 13, proving "$\Longleftarrow$" is enough, and the case $(r+s)(p^2-p)=0$ can again be isolated (cf. (39)), so $r \neq -s$.

*Case* 1: $r \neq s$.

(i) From Definition 14 and (13) we obtain

$$(\log f_{M_{r,s}}(t))' = \frac{1}{(s-r)t^2} \cdot (j_t(s) - j_t(r)) \qquad (t \in \mathbb{R}_+^*) \tag{40}$$

where $j_t: \mathbb{R} \to \mathbb{R}$,

$$j_t(v) := \frac{tv \cdot e^{tv}}{e^{tv}+1} - \log(e^{tv}+1) \qquad (v \in \mathbb{R};\ t \in \mathbb{R}_+^* \text{ fixed}).$$

Simple computations show that for every $t \in \mathbb{R}_+^*$, the function $j_t$ is even and $(\frac{d}{dv} j_t)(v) = \frac{t^2 v e^{tv}}{(e^{tv}+1)^2}$ $(v \in \mathbb{R})$, i.e. $j_t$ is strictly increasing on $\mathbb{R}_+^*$. Hence

$$j_t(s) - j_t(r) \gtrless 0 \quad (t \in \mathbb{R}_+^*) \quad \begin{pmatrix} |r| < |s| \\ |s| < |r| \end{pmatrix}$$

(notice that the case $r = -s$ is excluded), i.e. $\frac{1}{s^2 - r^2} \cdot (j_t(s) - j_t(r)) > 0$ $(t \in \mathbb{R}_+^*)$, i.e., by (40), $\frac{1}{r+s} \cdot (\log f_{M_{r,s}}(t))' > 0$ $(t \in \mathbb{R}_+^*)$, and finally by Theorem 8

$$\text{all } \varphi_p (p > 1) \text{ are strictly } \left\{ \begin{array}{l} M_{r,s}\text{-convex} \\ M_{r,s}\text{-concave} \end{array} \right. \text{ on } \mathbb{R}_+^* \ (r + s \gtrless 0). \tag{41}$$

(ii) For $t \in \mathbb{R}_+^*$, $\cosh tr \lessgtr \cosh ts$ $\quad \begin{pmatrix} |r| < |s| \\ |s| < |r| \end{pmatrix}$ and putting $z := e^t$, we have

$$z^r + z^{-r} \lessgtr z^s + z^{-s} \qquad (z > 1) \qquad \begin{pmatrix} |r| < |s| \\ |s| < |r| \end{pmatrix}. \tag{42}$$

Let be $x, y \in \mathbb{R}_+^*$, $x \neq y$. Then $z := \max\left\{\frac{x}{y}, \frac{y}{x}\right\} > 1$, and (42) implies

$$(x^r + y^r)(x^{-r} + y^{-r}) \lessgtr (x^s + y^s)(x^{-s} + y^{-s}) \quad \begin{pmatrix} |r| < |s| \\ |s| < |r| \end{pmatrix},$$

i.e.,

$$1 \lessgtr \frac{x^s + y^s}{x^r + y^r} \cdot \frac{x^{-s} + y^{-s}}{x^{-r} + y^{-r}} \quad \begin{pmatrix} |r| < |s| \\ |s| < |r| \end{pmatrix}.$$

Raising to the power with exponent $\frac{1}{s-r} \gtrless 0 \begin{pmatrix} r < s \\ s < r \end{pmatrix}$ and using Definition 14 leads to

$$1 \lessgtr M_{r,s}(x,y) \cdot M_{r,s}(x^{-1}, y^{-1}) \quad \left(r + s \gtrless 0\right),$$

i.e.,

$$\varphi_{-1} \text{ is strictly } \begin{array}{l} M_{r,s}\text{-convex} \\ M_{r,s}\text{-concave} \end{array} \text{ on } \mathbb{R}_+^* \qquad \left(r + s \gtrless 0\right). \tag{43}$$

(iii) In Case 1, the assertion follows from (41), (43), Theorem 6. 1° and 2°.

*Case 2:* $\quad r = s$.

(i) From Definition (14) and (13) $f_{M_{r,r}}(t) = e^{\frac{e^{tr}}{(e^{tr}+1)}}$ $(t \in \mathbb{R}_+^*)$. This is strictly $\begin{array}{l} \text{increasing} \\ \text{decreasing} \end{array}$ on $\mathbb{R}_+^*$ $\left(r \gtrless 0\right)$, and since $r + s = 2r$, (41) holds again.

(ii) Let be $x, y \in \mathbb{R}_+^*$, $x \neq y$. Then

$$M_{r,r}(x,y) \cdot M_{r,r}(x^{-1}, y^{-1}) \gtrless 1 \Longleftrightarrow \left(\frac{x}{y}\right)^{x^r - y^r} \gtrless 1. \tag{44}$$

On the other hand, $\left(\frac{x}{y}\right)^{x^r - x^r} \gtrless 1 \quad \left(r \gtrless 0\right)$, i.e., by (44)

$$M_{r,r}(x,y) \cdot M_{r,r}(x^{-1}, y^{-1}) \gtrless 1 \quad \left(r \gtrless 0\right).$$

Since $r + s = 2r$, (43) holds again, and the proof ends as in Case 1. □

**Remark 17.** By (27), (28), (38), (39)

$$E_{r,2r} = M_r = M_{r,0} \qquad \text{for all } r \in \mathbb{R},$$

and now (14) appears as a very special case of both Theorems 13 and 16.

Now we turn to a family of means which contains the limiting cases *min* and *max* of the family $(M_c)_{c \in \mathbb{R}}$ of power means and of other 1-parameter families of means (cf., e.g., [1], p. 346, for the generalized logarithmic means). The convexity/concavity situation of the power functions will be completely different from that of Theorems 13 and 16 (cf. Theorem 20). We begin by some characterization results.

**Lemma 18.** *Let $M$ be a homogeneous mean on $\mathbb{R}_+^*$ and*

$$g_M : \mathbb{R}^* \to \mathbb{R}_+^*, \; g_M(t) := (M(e^t, 1))^{\frac{1}{t}} \quad (t \in \mathbb{R}^*). \tag{45}$$

*Then we have:*

1°. *For every $c \in [0,1]$*

$$g_M(t) = e^c \text{ for all } t \in \begin{matrix} \mathbb{R}_+^* \\ \mathbb{R}_-^* \\ \mathbb{R}^* \end{matrix} \Leftrightarrow M(x,y) = x^c y^{1-c} \text{ for all } x,y \in \mathbb{R}_+^*, \quad \begin{matrix} x \geq y \\ x \leq y \\ \text{arbitrary} \end{matrix}.$$

2°. *If $M \in \mathcal{M}(\mathbb{R}_+^*)$, then for every $c \in [0,1]$*

$$g_M(t) = e^c \text{ for all } t \in \begin{matrix} \mathbb{R}_+^* \\ \mathbb{R}_-^* \end{matrix} \Leftrightarrow M(x,y) = \left\{ \begin{matrix} (\max\{x,y\})^c \cdot (\min\{x,y\})^{1-c} \\ (\min\{x,y\})^c \cdot (\max\{x,y\})^{1-c} \end{matrix} \right.$$

*for all $x, y \in \mathbb{R}_+^*$.*

3°. *If* $M \in \mathcal{M}(\mathbb{R}_+^*)$, *then*

$$g_M \text{ is constant on } \mathbb{R}^* \iff M = G.$$

*Proof.* By (45), we have for every $t \in \mathbb{R}^*$

$$g_M(t) = e^c \iff M(e^t, 1) = e^{ct}, \tag{46}$$

so, by (2), necessarily $c \in [0,1]$. Conversely, for $c \in [0,1]$, the expressions for $M(x,y)$ in 1°, 2°, 3° do provide homogeneous means on $\mathbb{R}_+^*$, namely weighted geometric means; the ones in 1° need not be symmetric.

1°. $\Longleftarrow$ immediately follows from (46).

$\Longrightarrow$: From the assumption and (46) and from $M(e^0, 1) = e^{c0}$ we obtain

$$M(e^t, 1) = e^{ct} \text{ for all } t \in \begin{matrix} \mathbb{R}_+ \\ \mathbb{R}_- \\ \mathbb{R} \end{matrix}. \tag{47}$$

Now let be $x, y \in \mathbb{R}_+^*$, $\begin{matrix} x \geq y \\ x \leq y \\ \textit{arbitrary} \end{matrix}$.

For $t := \log \frac{x}{y}$, the homogeneity of $M$ and (47) yield

$$\frac{1}{y} M(x,y) = M\left(\frac{x}{y}, 1\right) = \left(\frac{x}{y}\right)^c \quad \text{for all } x, y \in \mathbb{R}_+^*, \quad \begin{matrix} x \geq y \\ x \leq y \\ \textit{arbitrary} \end{matrix}$$

2°. $N_c(x,y) := (\max\{x,y\})^c \cdot (\min\{x,y\})^{1-c}$, for all $x, y \in \mathbb{R}_+^*$. Then

$$x^c y^{1-c} = \begin{cases} N_c(x,y) & x \geq y \\ N_{1-c}(x,y) & x \leq y \end{cases},$$

and by 1° and symmetry of $M, N_c, N_{1-c}$

$$g_M(t) = e^c \text{ for all } t \in \begin{matrix} \mathbb{R}_+^* \\ \mathbb{R}_-^* \end{matrix} \iff M(x,y) = \begin{cases} N_c(x,y) \\ N_{1-c}(x,y) \end{cases} \text{ for all } x, y \in \mathbb{R}_+^*, \begin{matrix} x \geq y \\ x \leq y \end{matrix}$$

$$\iff M(x,y) = \begin{cases} N_c(x,y) \\ N_{1-c}(x,y) \end{cases} \text{ for all } x, y \in \mathbb{R}_+^*.$$

3°. $\Longleftarrow$ is immediate from 1°.

$\Longrightarrow$: By 1° (lowest case) there exists $c \in [0,1]$ such that $M(x,y) = x^c y^{1-c}$ for all $\mathbb{R}_+^*$. Symmetry of $M$ implies $c = 1 - c$, i.e., $c = \frac{1}{2}$, i.e., $M = G$. □

**Corollary 19.** *For $M \in \mathcal{M}(\mathbb{R}_+^*)$, the function $\varphi_{-1}$ is $M$-affine if and only if $M = G$.*

*Proof.* $\varphi_{-1}$ is certainly $G - affine$. Conversely, by hypothesis $(M(x,y))^{-1} = M(x^{-1}, y^{-1})$ $(x, y \in \mathbb{R}_+^*)$, so $(M(x,x^{-1}))^{-1} = M(x^{-1},x) = M(x,x^{-1})$. Since $M(x,x^{-1}) \in \mathbb{R}_+^*$, we get $M(x,x^{-1}) = 1$ $(x \in \mathbb{R}_+^*)$. If $t \in \mathbb{R}^*$, $g_M(t) \overset{(45)}{=} (M(e^t,1))^{\frac{1}{t}} = (e^{\frac{t}{2}} \cdot M(e^{\frac{t}{2}}, e^{-\frac{t}{2}}))^{\frac{1}{t}} = (e^{\frac{t}{2}} \cdot 1)^{\frac{1}{t}} = e^{\frac{1}{2}}$, and by Lemma 18.3°, $M = G$. □

**Theorem 20.** *If $c \in [0,1]$ and $N_c(x,y) := (\max\{x,y\})^c \cdot (\min\{x,y\})^{1-c}$ $(x, y \in \mathbb{R}_+^*)$, then*

1°. $p \in \mathbb{R}_+ \Longrightarrow \varphi_p$ *is $N_c$-affine on $\mathbb{R}_+^*$.*

$$2^\circ.\ p \in \mathbb{R}_-^* \Longrightarrow \left[\varphi_p \ \textit{is} \ \begin{array}{c} \textit{strictly } N_c\textit{-convex} \\ N_c\textit{-affine} \\ \textit{strictly } N_c\textit{-concave} \end{array} \ \textit{on } \mathbb{R}_+^* \Longleftrightarrow c \begin{array}{c} > \\ = \\ < \end{array} \tfrac{1}{2}\right].$$

*Proof.* 1°. By Lemma 18.2°, $g_{N_c}$ is constant on $\mathbb{R}_+^*$. Since $N_c \in \mathcal{M}(\mathbb{R}_+^*)$, all $\varphi_p$ $(p > 1)$ are $N_c$-affine on $\mathbb{R}_+^*$ by Theorem 8, and by Theorem 6.1° and 4° so are all $\varphi_p$ $(0 < p < 1)$ as well as $\varphi_0$ and $\varphi_1$. (There is also a simple autonomous proof for 1°).

2°. Let be $x, y \in \mathbb{R}_+^*$ arbitrary, $x \neq y$. The fact $\max\{x^{-1},y^{-1}\}=(\min\{x,y\})^{-1}$ easily leads to

$$(N_c(x,y))^{-1} \begin{array}{c} < \\ = \\ > \end{array} N_c(x^{-1},y^{-1}) \Leftrightarrow (\min\{x,y\}^{2c-1} \begin{array}{c} < \\ = \\ > \end{array} (\max\{x,y\})^{2c-1} \Leftrightarrow c \begin{array}{c} > \\ = \\ < \end{array} \frac{1}{2}$$

which says that the assertion holds for $p = -1$. The general assertion now follows from 1° and Theorem 6.3°. □

**Remark 21.** Theorem 20 shows that in Theorem 6.2° and 3°, the hypothesis on $\varphi_{-1}$ is essential: All $\varphi_p$ $(p > 1)$ are $\begin{array}{c} N_1\text{-concave} \\ N_0\text{-convex} \end{array}$ on $\mathbb{R}_+^*$, but no $\varphi_q (q < 0)$ is $\begin{array}{c} N_1\text{-concave} \\ N_0\text{-convex} \end{array}$ on $\mathbb{R}_+^*$.

Also, strict convexity/concavity of all $\varphi_q (q < 0)$ does not imply that of all $\varphi_p$ $(p > 1)$.

## 5. Miscellaneous remarks and comments

**Remark 22.** For $M \in \mathcal{M}(\mathbb{R}_+^*)$, the function $\varphi_{-1}$ is $M$-convex on $\mathbb{R}_+^*$ if, and only if,

$$\sqrt{t} \leq M(t,1) \quad \text{for all} \quad t > 1$$

(see Corollary 19 for a related result). In fact: $(M(x,y))^{-1} \leq M(x^{-1}, y^{-1})$ $(x, y \in \mathbb{R}_+^*)$ is equivalent to $(M(t,1))^{-1} \leq M(t^{-1}, 1)$ $(t > 1)$ (see the procedure in the proof of Theorem 8), and this latter is equivalent to $1 \leq M(t,1) \cdot M(t^{-1}, 1)$ $(t > 1)$, $t \leq M(t,1) \cdot M(1,t)$ $(t > 1)$ and finally to $t \leq (M(t,1))^2$ $(t > 1)$.

**Remark 23.** The fact $G(x,y) < L(x,y) < A(x,y)$ $(x, y \in \mathbb{R}_+^*,\ x \neq y)$ is well-known (cf., e.g.[1], p. 348). In the light of Remark 3, this can be interpreted as the strict $L$-convexity of $\varphi_{-1}$ and $\varphi_2$ on $\mathbb{R}_+^*$, the latter being guaranteed by Theorem 9.2°. In fact: For $x, y \in \mathbb{R}_+^*, x \neq y$, $(G(x,y))^2 < (L(x,y))^2$ is equivalent to $(L(x,y))^{-1} < L(x^{-1}, y^{-1})$, and $(L(x,y))^2 < A(x,y) \cdot L(x,y)$ is equivalent to $(L(x,y))^2 < L(x^2, y^2)$.

**Remark 24.** All means $M \in \mathcal{M}(\mathbb{R}_+^*)$ occurring in Section 4 are continuous in each variable (notice that the contraharmonic mean $M_{1,2}$, in the sense of Definition 14, is not monotonic in $x$ and in $y$; cf. [2], bottom of p. 604). It then turned out that every $\varphi_p$ $(p \in \mathbb{R})$ is convex on $\mathbb{R}_+^*$ and/or concave on $\mathbb{R}_+^*$ (see Theorems 9, 10, 13, 16, 20). This is not so in general (a more general discussion will follow in a forthcoming paper).

**Example 25.** For $M(x,y) := \min\{x,y\} \cdot \left[\frac{\max\{x,y\}}{\min\{x,y\}}\right]$ $(x, y \in \mathbb{R}_+^*)$, we have $M \in \mathcal{M}$ $(\mathbb{R}_+^*)$ and $f_M(t) = [e^t]^{\frac{1}{t}}$ $(t \in \mathbb{R}_+^*)$, so $f_M$ is not monotonic on $\mathbb{R}_+^*$. Accordingly, by Theroem 8, $\varphi_{\frac{3}{2}}$ is neither $M$-convex nor $M$-concave on $\mathbb{R}_+^*:\ M(1^{\frac{2}{3}}, (2^{\frac{2}{3}})^{\frac{3}{2}}) = 2 > 1 = (M(1, 2^{\frac{2}{3}}))^{\frac{3}{2}}$, while $M(1^{\frac{3}{2}}, 2^{\frac{3}{2}}) = 2 < 2^{\frac{3}{2}} = (M(1,2))^{\frac{3}{2}}$. Furthermore, $\varphi_{-1}$ has the same behavior: $M(2,3) \cdot M(2^{-1}, 3^{-1}) = \frac{2}{3} < 1$, but $M(1,2) \cdot M(1^{-1}, 2^{-1}) = 2 > 1$.

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