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BANACH TYPE FIXED POINT THEOREMS ON PRODUCT OF SPACES

JANUSZ MATKOWSKI AND SHYAM LAL SINGH

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The main purpose of this paper is to introduce Banach operators on a finite product of metric spaces and obtain fixed point theorems for such operators.

1. Introduction

Let (Y, d) be a metric space. A map $T : Y \to Y$ is said to be a *Banach operator* if there exists a nonnegative number $q < 1$ such that $d(T^2x, Tx) \leq q\, d(Tx, x)$ for all x in Y. (cf. [1], [11], [13]). T is called *Banach contraction* (or simply a *contraction* in this paper) if there exists a nonnegative number $q < 1$ such that $d(Tx, Ty) \leq q\, d(x, y)$ for all $x, y \in Y$, and if $q = 1$ then T is *nonexpansive*. Evidently a contraction is a Banach operator. A Banach operator T is more general than a map $T : Y \to Y$ satisfying

$$d(Tx, Ty) \leq q \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \qquad (1.1)$$

for all $x, y \in Y$ and some nonnegative $q < 1$, (cf. [6], [7]). (The condition (1.1) is (21′) in Rhoades [12]). Indeed, a Banach operator has many peculiar properties. A Banach operator need not be continuous and it may have more than one fixed point. For example, $Tx = 0$ for $0 \leq x < 1/2$ and $Tx = 4/5$ for $1/2 \leq x \leq 1$. Evidently T is not a contraction but a discontinuous Banach operator with fixed points 0 and 4/5. A Banach operator may be even nonexpansive. For example, if T is the identity map on Y then it is a Banach operator and nonexpansive both. A discontinuous Banach operator need not have a fixed point. For further analysis and applications of Banach operators on various settings, one may refer to [1]–[3], [5]–[7], [11], [13], [15, p. 144] and [16]. In fact, the concept of a Banach operator is a variant of a condition

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essentially introduced by Cheney and Goldstein [3], and subsequently used by Cain, Jr. and Nashed [2], Taylor [16] and others.

THEOREM 1 *A continuous Banach operator T on a complete metric space* (Y, d) *has a fixed point.*

For its proof (indeed on more general settings) one may refer to Hicks and Rhoades [7] and Naimpally et al. [11]. (See also Corollary.)

Matkowski [9]–[10] (see also Kuczma et al. [8], and [14]) extended the concept of Banach contraction to a system of equations on a finite product of metric spaces and obtained a fixed point theorem for such a system of operators (cf. Theorem 2). The intent of this paper is to introduce Banach operators for a system of equations on a product of metric spaces and obtain some fixed point theorems for such operators.

2. Preliminaries

In all that follows, we generally follow the notations of Matkowski [9]–[10] (see also [8] and [14]).

Let a_{ik} be nonnegative numbers, $i, k = 1, \cdots, n$, and $c_{ik}^{(t)}$ square matrices defined in the following recursive manner:

$$c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for} \quad i \neq k \\ 1 - a_{ik} & \text{for} \quad i = k. \end{cases}, \qquad i, k = 1, 2, \cdots, n, \tag{2.1}$$

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)} & \text{for} \quad i \neq k \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)} & \text{for} \quad i = k \end{cases} \tag{2.2}$$

$i, k = 1, \cdots, n - t - 1$, $t = 0, 1, \cdots, n - 2$. If $n = 1$, we define $c_{11}^{(0)} = a_{11}$.

In [10] it is shown that the system of inequalities

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, \; i = 1, 2, \cdots, n,$$

has a solution $r_i > 0$, $i = 1, 2, \cdots, n$, if and only if,

$$c_{ii}^{(t)} > 0, \quad i = 1, \cdots, n - t, \; t = 0, \cdots, n - 1; \; n \geq 2. \tag{2.3}$$

Indeed, there exists a positive number $h < 1$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \le h\, r_i, \quad i = 1, 2, \cdots, n, \tag{2.4}$$

for some positive numbers $r_1, r_2, \cdots, r_n$, [10] (see also [8], [14]). Such an h may be found by

$$h = \max_{i} \left(r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right). \tag{2.5}$$

Let (X_i, d_i), $i = 1, 2, \cdots, n$, be metric spaces. Put

$$X := X_1 \times X_2 \times \cdots \times X_n.$$

Thus, in all that follows, $x \in X$ will mean $x = (x_1, \cdots, x_n)$.

Let us quote

THEOREM 2 ([9]–[10]) *Let X_i, $i = 1, \cdots, n$, be complete metric spaces and $T_i : X \to X_i$, $i = 1, \cdots, n$, be such that*

$$d_i(T_i x, T_i y) \le \sum_{k=1}^{n} a_{ik}\, d_k(x_k, y_k), \quad i = 1, \cdots, n, \tag{2.6}$$

for every $x_k, y_k \in X_k$, $k = 1, \cdots, n$, where a_{ik} are nonnegative numbers such that the matrices defined in (2.1) and (2.2) satisfy the condition (2.3). Then the system of equations

$$T_i x = x_i, \quad i = 1, \cdots, n, \tag{2.7}$$

has exactly one solution $p := (p_1, \cdots, p_n) \in X$. For an arbitrarily fixed $\overset{0}{x} = (\overset{0}{x_1}, \cdots, \overset{0}{x_n}) \in X$, the sequence of successive approximations

$$\overset{m+1}{x_i} = T_i(\overset{m}{x_1}, \cdots, \overset{m}{x_n}), \quad i = 1, \cdots, n, \quad m = 0, 1, \cdots, \tag{2.8}$$

converges and

$$p_i = \lim_{m \to \infty} \overset{m}{x_i}, \quad i = 1, \cdots, n. \tag{2.9}$$

System of maps $(T_1, \cdots, T_n)$ on X with values in metric spaces X_i, $i = 1, \cdots, n$, satisfying (2.6) may be called *system of contractions on a product of metric spaces* (or simply *contraction on product of spaces*), wherein (2.1)–(2.3) hold.

3. Banach operators on product spaces

Let $T_i : X \to X_i$, $i = 1, \cdots, n$. Then $T := (T_1, \cdots, T_n)$ satisfying

$$d_i\,(T_i(T_1x, T_2x, \cdots, T_nx),\ T_ix) \le \sum_{k=1}^{n} a_{ik}\, d_k\,(T_kx, x_k), \quad i = 1, \cdots, n, \tag{3.1}$$

for all $x \in X$, will be called a *system of Banach operators on product of spaces* (or simply Banach operators on product of spaces), if a_{ik} are nonnegative numbers such that the matrices defined in (2.1)–(2.2) have the properties (2.3). Such operators are natural generalization of contractions on product spaces (see Proposition below), and have porperties akin to its elder cousin "Banach operator on metric space". (See also Examples 1–2 below).

Consider the following condition for $T_i : X \to X_i$, $i = 1, \cdots, n$, which is much more general than (2.6) :

$$d_i\,(T_ix, T_i\,y) \le \tag{3.2}$$

$$\max\left\{\sum_{k=1}^{n} a_{ik}\, d_k\,(x_k, y_k),\ b \max\left\{ d_i\,(x_i, T_ix),\ d_i\,(y_i, T_i\,y),\ \frac{d_i\,(x_i, T_i\,y + d_i\,(y_i, T_i\,x)}{2}\right\}\right\}$$

for every $x, y \in X$, $i = 1, 2, \cdots, n$, where b and a_{ik} are nonnegative numbers such that $b < 1$ and (2.2)–(2.3) hold.

We remark that (3.2) with $b = 0$ is (2.6), and that Theorem 2 with (2.6) replaced by (3.2) remains true (see **[14]**). The single-valued version of the main result of Czerwik **[4]** is a special case of Theorem 2 with (2.6) replaced by (3.2). In particular, single-valued version of the condition involving a system of transformations for the main theorem[4] is included in (3.2).

PROPOSITION. *The following implications are true*:

(a) (2.6) *implies* (3.2) *and* (2.6) *implies* (3.1);

(b) (3.2) *implies* (3.1).

PROOF. (a): It follows from the preceding remark and (b).

(b): Let $x, y \in X$, and $y = T_i x$, i.e., $(y_1, \cdots, y_n) = (T_1 x, \cdots, T_n x)$. In particular $y_k = T_k x$. Then setting $y = Tx$ in (3.2) yields (3.1).

4. Main results

THEOREM 3 *Let* $(X_i, d_i), i = 1, \cdots, n$, *be complete metric spaces and* $T_i : X \to X_i$, $i = 1, \cdots, n$, *be a system of Banach operators on* X, *i.e.* (3.1) *and* (2.1)–(2.3) *hold. If* $T_1, \cdots, T_n$ *are continuous, then the system of equations* (2.7) *has a solution* $p = (p_1, \cdots, p_n)$, $p_i \in X_i$, $i = 1, \cdots, n$. *Further, there exists a point* x^0 *in* X *such that the sequence of successive approximations* (2.8) *converges and* (2.9) *holds.*

PROOF. Let $x^0 \in X$. Construct a sequence $\{x_i^m\}$, $i = 1, \cdots, n, m = 0, 1 \cdots$, by $x_i^{m+1} = T_i x^m$. We may assume (without any loss of generality **[10]**) that $d_i(x_i^1, x_i^0) \le r_i$, $i = 1, \cdots, n$, and $x_i^{m+1} \ne x_i^m$, $i = 1, \cdots, n$, (since otherwise $x_i^m = T_i x$, and the theorem is proved). From (3.1),

$$d_i(x_i^2, x_i^1) = d_i(T_i x^1, T_i x^0) = d_i(T_i(T_1 x^0, T_2 x^0, \cdots, T_n x^0), T_i x^0) \le$$

$$\sum_{k=1}^{n} a_{ik} d_k(T_k x^0, x_k^0) = \sum_{k=1}^{n} a_{ik} d_k(x_k^1, x_k^0) \le \sum_{k=1}^{n} a_{ik} r_k \le h r_i .$$

Similarly, $d_i(x_i^3, x_i^2) \le h(h r_i) = h^2 r_i$. Inductively, $d_i(x_i^{m+1}, x_i^m) \le h^m r_i$. This implies that $\{x_i^m\}$ is a Cauchy sequence which converges to some point $p_i \in X_i$, $(i = 1, \cdots, n)$. Using the continuity of T_i, we have that $T_i(p_1, \cdots, p_n) = p_i$, $i = 1, \cdots, n$. This completes the proof.

THEOREM 4 *Theorem* 3 *with the matrix* (a_{ik}) *symmetrical (which means,* $a_{ik} = a_{ki}$, $i, k = 1, \cdots, n$) *is equivalent to Theorem* 1 *with* $Y = X$, $T = (T_1, \cdots, T_n)$, *and* $d : X \times X \to R_+$ *(nonnegative reals) defined by*

$$d(x, y) = \sum_{i=1}^{n} r_i d_i(x_i, y_i), \quad x = (x_1, \cdots, x_n), \quad y = (y_1, \cdots, y_n) \in X.$$

In particular, (3.1) *reduces to a Banach operator (on* X*) whenever* (a_{ik}) *is a symmetrical matrix.*

PROOF. The metric space (X, d) is complete. Define a map $T : X \to X$ by $Tx = (T_1 x, \cdots, T_n x)$. We shall show that T is a Banach operator on (X, d). Note that

$$T^2x = T(Tx) = T(T_1x, \cdots, T_nx) =$$

$$(T_1(T_1x, \cdots, T_nx), T_2(T_1x, \cdots, T_nx), \cdots, T_n(T_1x, \cdots, T_nx)).$$

Since T_i, $i = 1, \cdots, n$, satisfy (3.1), we have for any $x \in X$,

$$d(T^2x, Tx) = \sum_{i=1}^{1} r_i\, d_i(T_i(T_1x, T_2x, \cdots, T_nx), T_ix)$$

$$\leq \sum_{i=1}^{n} r_i \sum_{k=1}^{n} a_{ik}\, d_k(T_kx, x_k) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ik}\, r_i \right) d_k(T_kx, x_k) .$$

By the symmetry of (a_{ik}) and (2.4),

$$\sum_{i=1}^{n} a_{ik}\, r_i = \sum_{i=1}^{n} a_{ik}\, r_i \leq h\, r_k.$$

Hence

$$d(T^2x, Tx) \leq \sum_{k=1}^{n} (h\, r_k)\, d_k(T_kx, x_k) = h\, d(Tx, x) .$$

This completes the proof since $0 < h < 1$.

COROLLARY. *Theorem* 1.

PROOF. Take $(Y, d) = (X_i, d_i)$, $T = T_i$, $i = 1, \cdots, n$, and $n = 1$ with $a_{11} = q$ in Theorem 3.

In the following examples, conditions of Theorem 2 are not satisfied but Theorem 3 is applicable.

EXAMPLE 1 Let $X_1 = X_2 = [0, 1]$ be metric spaces with usual metric, $X = X_1 \times X_2$ and

$$T_ix := T_i(x_1, x_2) = \begin{cases} \dfrac{x_i}{4} & \text{for } x \in X \text{ with } x_1 = x_2 \neq 1 , \\[2ex] \dfrac{1}{8} & \text{for } x = (1, 1);\ i = 1, 2 . \end{cases}$$

It can be seen that $T := (T_1, T_2)$ is a Banach operator, i.e., it satisfies (3.1) for $n = 2$. Note that $p = (0, 0)$ is the fixed point of T. Evidently, T does not satisfy (2.6).

EXAMPLE 2 Let X_1, X_2 and X be as in the above example, and $T_i : X \to X_i$, $i = 1, 2$, be such that

$$T_1 x = \begin{cases} \dfrac{x_1}{4}, & (x_1, x_2) \in X, x_1 \neq 1 \\ \dfrac{1}{8}, & (x_1, x_2) \in X, x_1 = 1, \end{cases}$$

$$T_2 x = \begin{cases} 0, & x_1 \in X_1, \quad 0 \le x_2 < \dfrac{1}{2} \\ \dfrac{2}{3}, & x_1 \in X_1, \quad \dfrac{1}{2} \le x_2 \le 1. \end{cases}$$

Then $T := (T_1, T_2)$ satisfies (3.1) but not (2.6). Note that $p = (0, 1)$ and $(0, 2/3)$ are fixed points of T.

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Department of Mathematics
Technical University
Willowa-2
43-309 Bielsko-Biala
Poland

Gurukula Kangri University
Hardwar
India-249404

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