Aequationes Mathematicae 52 (1996) 260–283
University of Waterloo

0001-9054/96/030260-24 \$1.50 + 0.20/0

On a class of composite functional equations in a single variable

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Summary. We prove a general result on continuous functions of the type $f: (0, \infty) \to (0, \infty)$ which satisfy the functional equation

$$f(x[f(x)]^p) = (f(x))^{p+1},$$

where p is an arbitrary fixed real number. Applying this result we determine all continuous solutions $f: [0, \infty) \to [0, \infty)$ for $p > 0$, as well as all the continuous solutions $f: \mathbb{R} \to \mathbb{R}$ for a positive integer p.

For $p = 1$ this equation is relevant to a division model of population.

Introduction

A functional equation is of the composite type if an unknown function acts on a combination of the variable and the function itself. To find the general solution or the general continuous solution of such an equation is difficult mainly because of its nonlinearity. There are some important examples of functional equations which are of the form

$$f(x\beta(f(x))) = \alpha(f(x)), \tag{$*$}$$

where the functions α and β are given. This equation has an interesting property. Namely, the problem to find all bijective solutions f leads to the linear iterative functional equation

$$\phi(\alpha(x)) = \beta(x)\phi(x), \qquad \phi = f^{-1},$$

which has a well known theory (cf. Kuczma [4], Chapter II, also Kuczma, Choczewski, Ger [5]). However, if f is not bijective, the problem to determine all

AMS (1991) subject classification: 39B12.

Manuscript received March 17, 1994 and, in final form, September 8, 1995.

continuous solutions is nontrivial. A good example is, for instance, the following functional equation related to a division model of population (cf. Dhombres [2], p. 6.1):

$$f(xf(x)) = (f(x))^2.$$

In this paper we deal with the composite functional equation $(*)$ where α and β are power functions of the form

$$\alpha(t) = t^{p+1}, \qquad \beta(t) = t^p,$$

for a $p \in \mathbb{R}$, $p \neq 0$. A main result of this paper, Theorem 1, gives the general continuous solution defined on the open interval $(0, \infty)$ for $p \neq -2$. In section 3, applying Theorem 1, we prove Theorem 2 which describes all continuous solutions defined on the closed interval $[0, \infty)$ for $p > 0$. In the case $p = 1$ Theorem 2 coincides with a result presented by Dhombres in [2], p. 6.1. As a corollary we obtain all nonnegative continuous solutions defined on $\mathbb{R}$. In section 3, assuming that p is a positive integer, we determine the general continuous solution of the type $f: \mathbb{R} \to \mathbb{R}$. We also show that there are a lot of discontinuous solutions. In particular, for $p = 1$, each rational homogeneous function $f: [0, \infty) \to [0, \infty)$ with rational values satisfies this functional equation.

1. Continuous solutions defined on $(0, \infty)$

A basic result of this paper reads as follows.

THEOREM 1. *Let $p \in \mathbb{R}$, $p \neq 0$, be fixed.*

1°. *If $-2 \neq p \neq -1$ then a continuous function $f: (0, \infty) \to (0, \infty)$ satisfies the functional equation*

$$f(x[f(x)]^p) = (f(x))^{p+1}, \qquad x > 0, \tag{1}$$

if, and only if, there exist $a, b \in [0, +\infty]$, $a \leq b$, and $a \neq b$ if $a = 0$ or $b = \infty$, such that

$$f(x) = \begin{cases} x/a & (0 < x \leq a), \\ 1 & (a < x < b), \\ x/b & (x \geq b). \end{cases}$$

2°. *If* $p = -2$ *then every function described in part* 1° *is a solution of equation* (1). *Moreover, for every continuous function* $f_1 : [1, \infty) \to [1, \infty)$ *such that* $f_1(1) = 1$, *and*

$$x \to \frac{f_1(x)}{x} \text{ is increasing on } [1, \infty),$$

there exists a unique continuous solution f *of equation* (1) *such that* $f(x) = f_1(x)$ *for all* $x \geq 1$. *Furthermore, the function* f *is an increasing homeomorphic mapping of* $(0, \infty)$ *onto itself.*

3°. *If* $p = -1$ *then a continuous function* $f: (0, \infty) \to (0, \infty)$ *satisfies* (1) *if, and only if, there are* $a, b \in [0, \infty]$, $a \leq b$, *and* $a \neq b$ *if* $a = 0$ *or* $b = \infty$; *and continuous functions* $f_a : (0, a] \to (0, \infty)$, $f_b : [b, \infty) \to (0, \infty)$ *satisfying the conditions*

$$\frac{x}{b} \leq f_a(x) \leq \frac{x}{a}, \qquad x \in (0, a]; \qquad \frac{x}{b} \leq f_b(x) \leq \frac{x}{a}, \qquad x \in [b, \infty);$$

$$\lim_{x \to a-} f_a(x) = 1 = \lim_{x \to b+} f_b(x)$$

such that

$$f(x) = \begin{cases} f_a(x) & 0 \leq x < a \\ 1 & a \leq x \leq b. \\ f_b(x) & x > b \end{cases}$$

Proof. First we prove 1° and 2°, i.e. we assume that $p \neq -1$.
Define the functions $M, D: (0, \infty) \to (0, \infty)$ by

$$M(x) := x[f(x)]^p, \qquad D(x) := \frac{f(x)}{x}, \qquad x > 0. \tag{2}$$

We can write equation (1) in the form

$$D(M(x)) = D(x), \qquad x > 0. \tag{3}$$

If $M(x_1) = M(x_2)$ for some $x_1, x_2 > 0$, then, by (3), we get $D(x_1) = D(x_2)$, i.e.

$$\frac{f(x_1)}{x_1} = \frac{f(x_2)}{x_2},$$

or, equivalently,

$$\frac{x_1[f(x_1)]^p}{x_1^{p+1}} = \frac{x_2[f(x_2)]^p}{x_2^{p+1}}.$$

Since $M(x_1) = x_1[f(x_1)]^p = x_2[f(x_2)]^p = M(x_2)$, it follows that $x_1^{p+1} = x_2^{p+1}$. Now the assumption $p \neq -1$ implies that $x_1 = x_2$. Thus M is one-to-one and, consequently, strictly monotonic.

Suppose first that M is strictly increasing. Put

$$\operatorname{Fix}(M) := \{x > 0: M(x) = x\},$$

and note that

$$\operatorname{Fix}(M) = \{x > 0: f(x) = 1\}.$$

We shall prove that $\operatorname{Fix}(M)$ is a nonempty subinterval of $(0, \infty)$. For contradiction, suppose that $\operatorname{Fix}(M) = \varnothing$. The continuity of M implies either

$$M(x) < x, \qquad x > 0,$$

or

$$M(x) > x, \qquad x > 0.$$

In the first case, by the definition of M, we would have

$$\begin{aligned} &\text{for } p > 0: \quad f(x) < 1, \quad x > 0; \\ &\text{for } p < 0: \quad f(x) > 1, \quad x > 0. \end{aligned} \tag{4}$$

Since M and D are continuous, and M is strictly increasing, equation (3) implies that

$$D((0, \infty)) = D([M(1), 1]).$$

(Of course, instead of 1 we could take an arbitrary point x_0). It follows that the numbers

$$c := \inf D((0, \infty)), \qquad C := \sup D((0, \infty)),$$

are positive and finite. Hence $c \le D(x) \le C$ for all $x > 0$, i.e.

$$cx \le f(x) \le Cx, \qquad x > 0,$$

which contradicts (4). In the case $M(x) > x$ for all $x > 0$ the argument is analogous. Thus $\mathrm{Fix}(M) \ne \varnothing$.

Now we show that $\mathrm{Fix}(M)$ is an interval. For a proof by contradiction suppose that this is false. Then there would exist an interval $[c, d]$, $c < d$, such that $c, d \in \mathrm{Fix}(M)$, and $(c, d) \cap \mathrm{Fix}(M) = \varnothing$. Consequently, either $M(x) < x$ for all $x \in (c, d)$, or $M(x) > x$ for all $x \in (c, d)$. In the first case we would have

$$\lim_{n \to \infty} M^n(x) = c, \qquad x \in [c, d),$$

where M^n stands for the n-th iterate of the function M. From equation (3), by induction, we get

$$D(x) = D(M^n(x)), \qquad n = 0, 1, 2, \ldots; x \in (0, \infty),$$
$$n = -1, -2, \ldots; x \in M^{-n}((0, \infty)). \tag{5}$$

Therefore the continuity of D implies

$$D(x) = \lim_{n \to \infty} D(M^n(x)) = D(c), \qquad x \in [c, d).$$

Hence, again by the continuity of D, we get $D(c) = D(d)$, i.e.

$$f(c)d = f(d)c.$$

On the other hand we have $M(c) = c$ and $M(d) = d$, which means that

$$f(c) = 1, \qquad f(d) = 1.$$

The last two relations imply that $c = d$, which is the desired contradiction. If $M(x) > x$ for all $x \in (c, d)$ we can argue in the same way.

Put

$$a := \inf \mathrm{Fix}(M), \qquad b := \sup \mathrm{Fix}(M).$$

According to what we have proved,

$$0 \le a < +\infty, \qquad 0 < b \le +\infty.$$

Since M is continuous we have

$$\mathrm{Fix}(M) = [a, b] \cap (0, +\infty), \qquad 0 \le a \le b \le +\infty.$$

Hence,

$$f(x) = 1, \qquad x \in [a, b] \cap (0, +\infty). \tag{6}$$

Now we shall consider the following cases.

Case 1. $a = 0$ and $b = +\infty$.
Then (6) implies that

$$f(x) = 1, \qquad x > 0.$$

Case 2. $a = 0$ and $b < +\infty$.
In view of (6)

$$f(x) = 1, \qquad 0 < x \le b.$$

Moreover, we have either $M(x) < x$ for all $x > b$, or $M(x) > x$ for all $x > b$. Suppose first that $M(x) < x$ for all $x > b$. Then, for a fixed $x > b$,

$$\lim_{n \to \infty} M^n(x) = b.$$

Hence, by (5) and the continuity of D,

$$D(x) = \lim_{n \to \infty} D(M^n(x)) = D(b), \qquad x > b.$$

If $M(x) > x$ for all $x > b$, then

$$\lim_{n \to \infty} M^{-n}(x) = b, \qquad x > b,$$

and, for the same reason,

$$D(x) = \lim_{n \to \infty} D(M^{-n}(x)) = D(b), \qquad x > b.$$

Therefore in both cases

$$f(x) = b^{-1} f(b) x = b^{-1} x, \qquad x > b.$$

Thus we have proved that

$$f(x) = \begin{cases} 1 & 0 < x < b \\ x/b & x \geq b \end{cases}.$$

In the same way we obtain the form of the solution in the remaining two cases:

CASE 3. $a > 0$ and $b = +\infty$. Then

$$f(x) = \begin{cases} x/a & 0 < x \leq a \\ 1 & x > a \end{cases}.$$

CASE 4. $0 < a \leq b < +\infty$. Then

$$f(x) = \begin{cases} x/a & 0 < x \leq a. \\ 1 & a < x < b. \\ x/b & x \geq b \end{cases} \tag{7}$$

It is easy to verify that the functions given by the above formulas satisfy equation (1). Note also that, with obvious conventions, the formula (7) describes the general continuous solution of equation (1) in the considered general case when M is increasing.

Now suppose that M is strictly decreasing. Then the function $x \to [f(x)]^p$ is also strictly decreasing on $[0, \infty)$.

Hence for $p > 0$ the function f is strictly decreasing. This is a contradiction because the function $f \circ M$ on the left-hand side of equation (1) is strictly increasing and on the right-hand side, $x \to [f(x)]^{p+1}$, is decreasing on $[0, \infty)$.

For $p < 0$ the function f is strictly increasing. Let us consider two subcases: $p \in (-1, 0)$ and $p < -1$.

If $-1 < p < 0$ then $f \circ M$ is strictly decreasing and the function $(0, \infty) \ni x \to [f(x)]^{p+1}$ is increasing, which is a contradiction.

Now suppose that $p < -1$. For convenience put $r := -(p + 1)$. Then $r > 0$, and equation (1) can be written in the form

$$f\left(\frac{x}{[f(x)]^{r+1}}\right) = \frac{1}{[f(x)]^r}, \qquad x > 0. \tag{8}$$

Since M is decreasing, and

$$M(x) = \frac{x}{[f(x)]^{r+1}}, \qquad x > 0,$$

the function f must be an increasing homeomorphic map of $(0, \infty)$ onto itself. Hence, putting $\phi := f^{-1}$, one can write equation (8) in the equivalent form

$$\frac{\phi(x)}{x} = \frac{\phi(x^{-r})}{x^{-r}}, \qquad x > 0. \tag{9}$$

Suppose that $p \neq -2$. Then $r > 0$ and $r \neq 1$. Setting $\gamma(x) := x^{-1}\phi(x)$, $x > 0$, we can write equation (9) in the form $\gamma(x) = \gamma(x^{-r})$, $x > 0$. Iterating this functional equation, and making use of the continuity of γ at the point 1, it is easy to see that $\gamma(x) = \gamma(1)$ for all $x > 0$. Hence $\phi(x) = \phi(1)x$, and consequently, $f(x) = f(1)x$, $x > 0$. This completes the proof of 1°.

Now suppose that $p = -2$. Then $r = 1$, and equation (9) is of the form

$$\phi(x) = x^2\phi(x^{-1}), \qquad x > 0.$$

Take an arbitrary continuous function $f_1 : [1, \infty) \to [1, \infty)$ such that $f_1(1) = 1$, and

$$x \to \frac{f_1(x)}{x} \text{ is increasing on } [0, \infty).$$

Of course f_1 is strictly increasing and $f_1([1, \infty)) = [1, \infty)$. It follows that $\phi_1 := f_1^{-1}$ is strictly increasing, continuous, $\phi_1([1, \infty)) = [1, \infty)$, and the function

$$x \to \frac{\phi_1(x)}{x} \text{ is decreasing on } [0, \infty). \tag{10}$$

Define $\phi_0 := (0, 1] \to (0, \infty)$ by

$$\phi_0(x) := x^2\phi_1(x^{-1}), \qquad x \in (0, 1].$$

Of course ϕ_0 is continuous, $\phi_0(1) = \phi_1(1)$, and $\phi : (0, \infty) \to (0, \infty)$ defined by

$$\phi(x) = \begin{cases} \phi_0(x), & x \in (0, 1) \\ \phi_1(x), & x \in [1, \infty) \end{cases}$$

satisfies equation (9). By the definitions of ϕ, ϕ_0, and property (10), we have

$$\lim_{x\to 0+} \phi(x) = \lim_{x\to 0+} \phi_0(x) = \lim_{x\to 0+} x\frac{\phi_1(x^{-1})}{x^{-1}} = 0.$$

Take arbitrary $x, y \in (0, 1)$, $x < y$. Then $x^{-1} > y^{-1}$, and by (10) we have

$$\frac{\phi_1(x^{-1})}{x^{-1}} \le \frac{\phi_1(y^{-1})}{y^{-1}}.$$

Consequently,

$$\phi_0(x) = x\frac{\phi_1(x^{-1})}{x^{-1}} < y\frac{\phi_1(y^{-1})}{y^{-1}} = \phi_0(y).$$

Thus the function ϕ is an increasing homeomorphism of $(0, \infty)$. The function $f := \phi^{-1}$ is a solution of equation (1) and $f|_{[1,\infty)} = f_1$. This completes the proof of 2°.

Now we prove part 3°. Setting $p = -1$ in (1) gives

$$f\left(\frac{x}{f(x)}\right) = 1, \qquad x > 0. \tag{11}$$

Put

$$a := \inf\left\{\frac{x}{f(x)} : x > 0\right\}, \qquad b := \sup\left\{\frac{x}{f(x)} : x > 0\right\}.$$

Since f is continuous and positive, the range of the function $x \to x/f(x)$ is a nonempty interval I with the endpoints a and b which, of course, satisfy the conditions $a, b \in [0, \infty]$, $a \le b$, and $a \ne b$ if $a = 0$ or $b = \infty$. By (11) and the continuity of f we have $f(x) = 1$ for all $x \in [a, b] \cap (0, \infty)$. Denote by f_a and f_b, respectively, the restrictions of the function f to the intervals $(0, a]$ and $[b, \infty)$. By the definition of a and b we have

$$a \le \frac{x}{f_a(x)} \le b, \qquad x \in (0, a]; \qquad a \le \frac{x}{f_b(x)} \le b, \qquad x \in [b, \infty);$$

and, by the continuity of f,

$$\lim_{x\to a-} f_a(x) = 1 = \lim_{x\to b+} f_b(x).$$

Take now arbitrary $a, b \in [0, \infty]$, such that $a \leq b$, and $a \neq b$ if $a = 0$ or $b = \infty$, and two arbitrary continuous functions $f_a: (0, a] \to \mathbb{R}$, $f_b: [b, \infty) \to \mathbb{R}$ satisfying the above conditions. An easy verification shows that the function $f: (0, \infty) \to (0, \infty)$ such that $f|_{(0,a]} = f_a$, $f|_{[b,\infty)} = f_b$, $f|_{[a,b]} \equiv 1$, is a continuous solution of equation (11). This completes the proof.

REMARK 1. The above result gives the general continuous solution for all $p \neq -2$ such that $p \neq 0$. To explain the assumption $p \neq 0$ let us note that for $p = 0$ equation (1) becomes the identity $f \equiv f$. Thus every function $f: (0, \infty) \to (0, \infty)$ is a solution.

REMARK 2. The graphs of the general continuous solutions of equation (1) for $p \in \mathbb{R} \setminus \{0, -2\}$, and $p = -1$, are given by Figures 1 and 2, respectively. Note that if $a = b$ then, in both cases, the solution is a linear function $f(x) = f(1)x$, $x > 0$.

REMARK 3. Suppose that $p = -1$. Then $f: (0, \infty) \to (0, \infty)$ is a (not necessarily continuous) solution of equation (1) if, and only if, there exists a nonempty set $A \subset \mathbb{R}$ such that $f(x) = 1$ for all $x \in A$, and $x/f(x) \in A$ for all $x \in (0, \infty) \setminus A$.

2. Continuous solutions defined on $[0, \infty)$, and continuous nonnegative solutions defined on $\mathbb{R}$

We begin this section with a generalization of a result presented by Dhombres in [2], p. 6.1, where the case $p = 1$ was considered.

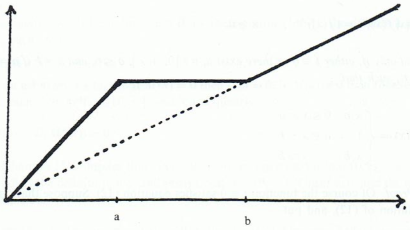

Figure 1. $p \in \mathbb{R} \setminus \{0, -2\}$.

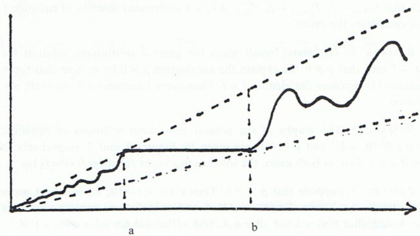

Figure 2. $p = -1$.

THEOREM 2. *Let $p > 0$ be fixed. Then a continuous function $f: [0, \infty) \to [0, \infty)$ satisfies the functional equation*

$$f(x[f(x)]^p) = (f(x))^{p+1}, \qquad x \geq 0, \tag{12}$$

if, and only if, either $f \equiv 0$ or there exist $a, b \in [0, +\infty]$, $a \leq b$, and $a \neq b$ if $a = 0$ or $b = \infty$, such that

$$f(x) := \begin{cases} x/a & 0 \leq x < a \\ 1 & a \leq x \leq b. \\ x/b & x > b \end{cases}$$

Proof. Of course the function $f \equiv 0$ satisfies equation (12). Suppose that $f \not\equiv 0$ is a solution of (12), and put

$$A := \{x \geq 0 : f(x) = 0\}.$$

First we prove the following

CLAIM. *Either $A = \varnothing$ or A is a closed interval.*

For contradiction, suppose that there exist $c, d \in A$, such that $c < d$ and $f(x) > 0$ for all $x \in (c, d)$. Define $M: [0, \infty) \to [0, \infty)$, and $D: (0, \infty) \to [0, \infty)$ by the formulas

$$M(x) := x[f(x)]^p, \qquad x \geq 0; \qquad D(x) := \frac{f(x)}{x}, \qquad x > 0,$$

and note that from equation (12) we obtain

$$D(M(x)) = D(x), \qquad x > 0.$$

Now, by the same argument as in the proof of Theorem 1, we can show that M is one-to-one on the interval (c, d). This is a contradiction because $M(c) = 0 = M(d)$. Thus the claim is proved.

If $A \subseteq \{0\}$ the result is an easy consequence of Theorem 1.

Suppose that there exists a $z \in A$ such that $z > 0$. Then

$$f(0) = f(z[f(z)]^p) = [f(z)]^{p+1} = 0,$$

which proves that also $0 \in A$. According to our Claim the set A is a closed interval. We are going to show that $A = [0, \infty)$. To obtain a contradiction suppose that

$$z_0 := \sup A < \infty.$$

Consequently, $A = [0, z_0]$. Since $p > 0$, by the continuity of M, there exists a point $c > z_0$ such that

$$M(x) < x, \qquad x \in (0, c].$$

Take an arbitrary $x \in (z_0, c]$. If $M(x) = 0$ then $f(x) = 0$. If $M(x) \neq 0$ then there exists $n \in \mathbb{N}$ such that $M^n(x) \in (0, z_0]$, and, consequently,

$$D(x) = D(M^n(x)) = 0.$$

The definition of D implies that $f(x) = 0$. Thus we have shown that $[0, c] \subset A$. This contradicts the definition of z_0 and proves that $A = [0, \infty)$. Thus $f \equiv 0$, and the proof is completed.

REMARK 4. The assumption $p > 0$ in the proof of Theorem 2 plays an important role. The problem to determine the continuous solutions $f: [0, \infty) \to [0, \infty)$ of equation (12) in the three remaining cases $p \in (-1, 0)$, $p = -1$, and $p \in (-\infty, -1)$ is open.

For arbitrary fixed $a, b \in \mathbb{R}$ such that $0 < a \leq b \leq +\infty$ we define a function $f_{a,b}: [0, \infty) \to [0, \infty)$ by the formula

$$f_{a,b}(x) := \begin{cases} x/a & 0 \leq x < a \\ 1 & a \leq x < b. \\ x/b & x \geq b \end{cases}$$

Now we shall prove

COROLLARY 1. *Let $p > 0$ be fixed. Then a continuous function $f: \mathbb{R} \to [0, \infty)$ satisfies the functional equation*

$$f(x[f(x)]^p) = (f(x))^{p+1}, \qquad x \in \mathbb{R},$$

if, and only if, f has one of the following forms:

1°. *there exist $a, b \in \mathbb{R}$, $-\infty \leq a < 0 < b \leq +\infty$, such that*

$$f(x) = \begin{cases} x/a & x < a \\ 1 & a \leq x < b; \\ x/b & x \geq b \end{cases}$$

2°. $f \equiv 0$;

3°. *there exist $a, b \in \mathbb{R}$, $0 < a \leq b \leq +\infty$, such that*

$$f(x) = \begin{cases} 0 & x < 0 \\ f_{a,b}(x) & x \geq 0 \end{cases}, \quad \text{or} \quad f(x) = \begin{cases} f_{a,b}(-x) & x < 0 \\ 0 & x \geq 0 \end{cases};$$

4°. *there exist $a, b, c, d \in \mathbb{R}$, $0 < a \leq b \leq +\infty$, $0 < c \leq d \leq +\infty$, such that*

$$f(x) = \begin{cases} f_{c,d}(-x) & x < 0 \\ f_{a,b}(x) & x \geq 0 \end{cases}.$$

Proof. This is a consequence of Theorem 2 and of the obvious fact that $f: \mathbb{R} \to [0, \infty)$ satisfies the considered equation iff the function $g: \mathbb{R} \to [0, \infty)$, $g(x) := f(-x)$, $x \in \mathbb{R}$, does.

3. Continuous solutions defined on $\mathbb{R}$ for $p \in \mathbb{N}$

In this section we show that Theorem 2 can be applied to find all the continuous solutions of the functional equation

$$f(x[f(x)]^n) = (f(x))^{n+1}, \qquad x \in \mathbb{R}, \tag{13}$$

where n is a fixed positive integer. It is convenient to consider separately the cases when n is odd and even. The case when n is odd is more interesting (for $n = 1$ we get the functional equations mentioned in the Introduction), and a little more difficult (for even n the function $x[f(x)]^n$ is nonnegative for $x \geq 0$ and nonpositive for $x \leq 0$, which simplifies the considerations). Therefore we give a detailed theory of the continuous solutions $f: \mathbb{R} \to \mathbb{R}$ of equation (13) where n is a fixed odd positive integer.

Let us make some obvious remarks.

REMARK 5. Let $n \in \mathbb{N}$ be fixed, and suppose that $f: \mathbb{R} \to \mathbb{R}$ is a solution of equation (13). If n is odd then either $f(0) = 0$, or $f(0) = 1$. If n is even then either $f(0) = 0$, $f(0) = 1$, or $f(0) = -1$.

REMARK 6. Let $n \in \mathbb{N}$ be fixed. If $f: \mathbb{R} \to \mathbb{R}$ is a solution of equation (13) and there exists a point $z_0 \in \mathbb{R}$ such that $f(z_0) = 0$, then $f(0) = 0$.

REMARK 7. Let $n \in \mathbb{N}$ be fixed. A function $f: \mathbb{R} \to \mathbb{R}$ satisfies (13) iff the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) := f(-x)$, $x \in \mathbb{R}$, does. Moreover, if n is even, then f satisfies (13) iff the function $(-f)$ does.

Let us also make the following obvious

REMARK 8. If $n \in \mathbb{N}$ is odd and $f: \mathbb{R} \to \mathbb{R}$ is a nonpositive solution of equation (13) then $f \equiv 0$.

We need the following

LEMMA 1. *Let $n \in \mathbb{N}$ be fixed. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a continuous solution of equation* (13) *and there exists a point $z \neq 0$ such that $f(z) = 0$.*

If $z < 0$ *then* $f(x) = 0$ *for all* $x \leq 0$.

If $z > 0$ *then* $f(x) = 0$ *for all* $x \geq 0$.

Proof. In view of Remark 7, without any loss of generality, we may assume that $z > 0$. Put $A := \{x \geq 0 : f(x) = 0\}$. The same reasoning as in the proof of the Claim shows that A is a closed interval. By Remark 6 we have $f(0) = 0$, and consequently, $0 \in A$. Put

$$z_0 := \sup A,$$

and suppose that $z_0 < \infty$. According to what we have already shown, $f(x) = 0$ for all $x \in [0, z_0]$. Moreover, either

$$f(x) > 0, \qquad x > z_0,$$

or

$$f(x) < 0, \qquad x > z_0.$$

In the first case, by the continuity of f and $f(z_0) = 0$, there would exist an $\varepsilon > 0$ such that $0 < x[f(x)]^n \leq z_0$ for all $x \in [z_0, z_0 + \varepsilon]$. Hence, from equation (13), we would have

$$(f(x))^{n+1} = f(x[f(x)]^n) = 0, \qquad x \in [z_0, z_0 + \varepsilon].$$

This contradicts the definition of z_0, and shows that $z_0 = \infty$.

Now consider the second case: $f(x) < 0$ for all $x > z_0$.

If n is even, then $x[f(x)]^n > 0$ for all $x > z_0$. The continuity of f and $f(z_0) = 0$, imply that there exists an $\varepsilon > 0$ such that $0 < x[f(x)]^n \leq z_0$ for all $x \in [z_0, z_0 + \varepsilon]$, and, by (13),

$$(f(x))^{n+1} = f(x[f(x)]^n) = 0, \qquad x \in [z_0, z_0 + \varepsilon].$$

This contradiction completes the proof for even n.

Let n be odd. Suppose first that there exists a $z < 0$ such that $f(z) = 0$. In the same way as in the previous case, we can show that $f(x) = 0$ for all $x \in (x_0, 0]$, where

$$x_0 := \inf\{z : f(z) = 0\} < 0.$$

We are going to show that $x_0 = -\infty$. For contradiction, suppose that $x_0 > -\infty$. Then we have either $f(x) < 0$ for all $x < x_0$, or $f(x) > 0$ for all $x > x_0$. By Remark 8 the first possibility cannot happen. Suppose that $f(x) > 0$ for all $x < x_0$. Then, in view of Remark 7, the function $g: \mathbb{R} \to \mathbb{R}$, given by $g(x) := f(-x)$, $x \in \mathbb{R}$, would be a continuous solution of equation (13), such that $g(x) = 0$ for all $x \in [0, -x_0]$, and $f(x) > 0$ for all $x > -x_0$. This is impossible as we have already shown in the first case. Thus $x_0 = -\infty$, and, consequently,

$$f(x) = 0, \qquad x \le z_0; \qquad f(x) < 0, \qquad x > z_0,$$

which contradicts Remark 8. Now suppose that $f(x) \neq 0$ for all $x < 0$. Then, by the continuity of f, either $f(x) < 0$ for all $x < 0$, or $f(x) > 0$ for all $x < 0$. By Remark 8 the first case cannot happen. In the second case the function $g: [0, \infty) \to [0, \infty)$ defined by

$$g(x) := f(-x), \qquad x \ge 0,$$

is a continuous solution of equation (12) where $p = n$. Since $g(0) = 0$, and g is positive on $(0, \infty)$, by Theorem 2 there is an $a > 0$ such that

$$g(x) = \frac{x}{a}, \qquad 0 \le x \le a,$$

and, consequently,

$$f(x) = -\frac{x}{a}, \qquad -a \le x \le 0.$$

Since $f(x_0) = 0$ and $f(x) < 0$ for all $x > x_0$, the continuity of f implies that there is an $\varepsilon > 0$ such that

$$x[f(x)]^n \in (-a, 0) \qquad \text{for all } x \in (x_0, x_0 + \varepsilon).$$

Therefore, making use of the functional equation (13), we get

$$-\frac{x[f(x)]^n}{a} = (f(x))^{n+1}, \qquad x \in (x_0, x_0 + \varepsilon),$$

i.e.

$$f(x) = -\frac{x}{a}, \qquad x \in (x_0, x_0 + \varepsilon).$$

Now the continuity of f implies that $f(x_0) = (-x_0/a)$ is negative. This contradiction completes the proof.

LEMMA 2. *Let $n = 2k - 1$, $k \in \mathbb{N}$, be fixed. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a continuous solution of equation* (13) *such that*

$$f(x) < 0, \qquad x < 0,$$

and

$$f(x) > 0, \qquad x > 0.$$

Then there exist $a, b \in (0, +\infty]$, $a \le b$, and $a \ne b$ if $b = \infty$, such that

$$f(x) = \begin{cases} -f_{a,b}(-x) & x < 0 \\ f_{a,b}(x) & x \ge 0 \end{cases}.$$

Proof. The continuity of f implies that $f(0) = 0$. Therefore, by Theorem 2, there exist $a, b \in (0, +\infty]$, $a \le b$, and $a \ne b$ if $b = \infty$, such that

$$f(x) = f_{a,b}(x), \qquad x \ge 0.$$

Since the function $M(x) := x[f(x)]^{2k-1}$ is one-to-one and continuous on $(-\infty, 0]$, it follows that M is strictly decreasing on $(-\infty, 0]$ and $M(0) = 0$. In particular $M(x) > 0$ for all $x < 0$, and, by equation (13), we have

$$(f(x))^{2k} = f_{a,b}(M(x)), \qquad x < 0. \tag{14}$$

Take now arbitrary $x < 0$. Note that, by the definition of $f_{a,b}$, we have

$$M(x) = x[f(x)]^{2k-1} \in [a, b) \;\Leftrightarrow\; (f(x))^{2k} = 1 \;\Leftrightarrow\; f(x) = -1,$$

so, by (13), $f(-x) = 1$, and, consequently, $x \in (-b, -a]$. Since $M(-a) = a$, the monotonicity of M implies that for all $x \in (-a, 0)$ we have $M(x) \in (0, a)$. Now (14) and the definition of $f_{a,b}$ give

$$(f(x))^{2k} = \frac{x[f(x)]^{2k-1}}{a}, \qquad x \in (-a, 0),$$

which means that $f(x) = a^{-1}x$ for all $x \in (-a, 0)$. Assuming that $b < \infty$, in the same way we prove that $f(x) = b^{-1}x$ for all $x \in (-\infty, -b)$. This completes the proof.

Now we can prove the main result of this section.

THEOREM 3. *Let $k \in \mathbb{N}$ be fixed. A function $f: \mathbb{R} \to \mathbb{R}$ is a continuous solution of the functional equation*

$$f(x[f(x)]^{2k-1}) = (f(x))^{2k}, \qquad x \in \mathbb{R},$$

if, and only if, f has one of the following forms:

1°. *there exist $a, b \in \mathbb{R}$, $-\infty \le a < 0 < b \le +\infty$, such that*

$$f(x) = \begin{cases} x/a & x < a \\ 1 & a \le x < b; \\ x/b & x \ge b \end{cases}$$

2°. $f \equiv 0$;

3°. *there exist $a, b \in \mathbb{R}$, $0 < a \le b \le +\infty$, such that*

$$f(x) = \begin{cases} 0 & x < 0 \\ f_{a,b}(x) & x \ge 0 \end{cases}, \quad \text{or} \quad f(x) = \begin{cases} f_{a,b}(-x) & x < 0 \\ 0 & x \ge 0 \end{cases};$$

4°. *there exist $a, b, c, d \in \mathbb{R}$, $0 < a \le b \le +\infty$, $0 < c \le d \le +\infty$, such that*

$$f(x) = \begin{cases} f_{c,d}(-x) & x < 0 \\ f_{a,b}(x) & x \ge 0 \end{cases};$$

5°. *there exist $a, b \in \mathbb{R}$, $0 < a \le b \le +\infty$, such that*

$$f(x) = \begin{cases} -f_{a,b}(-x) & x < 0 \\ f_{a,b}(x) & x \ge 0 \end{cases}, \quad \text{or} \quad f(x) = \begin{cases} f_{a,b}(x) & x < 0 \\ -f_{a,b}(-x) & x \ge 0 \end{cases}.$$

Proof. Suppose first that $f(x) \ne 0$ for all $x \in \mathbb{R}$. By Remark 8 the function f must be positive everywhere. In particular $f(0) = 1$. By Theorem 2 there exists a $b \in (0, \infty]$ such that $f(x) = 1$ for all $x \in [0, b)$, and $f(x) = b^{-1}x$ for all $x \ge b$. In the same way, making use of Remark 7, we show that there exists an $a \in [-\infty, 0)$

such that $f(x) = 1$ for all $x \in (a, 0]$, and $f(x) = a^{-1}x$ for all $x \leq a$. This shows that f is of the form 1°.

Suppose that there exists $z \in \mathbb{R}$, $z \neq 0$, such that $f(z) = 0$. If $z < 0$ then by Lemma 1 we have $f(x) = 0$ for all $x \leq 0$. By Remark 8 we have $f(x) \geq 0$ for all $x \geq 0$. Therefore, applying Theorem 2, we get that either $f \equiv 0$ or there exist $a, b \in \mathbb{R}$, $0 < a \leq b \leq +\infty$, such that $f(x) = f_{a,b}(x)$ for all $x \geq 0$, i.e. f is given by the first of the formulas 3°. If $z > 0$, making use of Remark 7 and applying the same reasoning as above, we show that either $f \equiv 0$, or f is given by the second formula in 3°.

Now suppose that $f(z) = 0$ iff $z = 0$. Then, in view of Remark 8, only one of the following three cases can occur:

(i) $f(x) > 0, \qquad x \in \mathbb{R}, \qquad x \neq 0;$

(ii) $f(x) < 0, \qquad x < 0, \quad$ and $\quad f(x) > 0, \qquad x > 0;$

(iii) $f(x) > 0, \qquad x < 0, \quad$ and $\quad f(x) < 0, \qquad x > 0.$

In case (i), applying Theorem 2 and Remark 7, it is easy to see that f must be of the form 4°.

In the case (ii), by Lemma 2, we get the first formula of 5°.

In the case (iii), by Remark 7 and Lemma 2, we get the second formula of 5°.

Since all the functions given in the formulas 1°–5° satisfy the considered functional equation, the proof is completed.

In a similar way, applying Remarks 5, 6, 7, and Lemma 1, we can prove the following:

THEOREM 4. *Let $k \in \mathbb{N}$ be fixed. A function $f: \mathbb{R} \to \mathbb{R}$ is a continuous solution of the functional equation*

$$f(x[f(x)]^{2k}) = (f(x))^{2k+1}, \qquad x \in \mathbb{R},$$

if, and only if, f has one of the following forms:

1°. $f \equiv 0$;

2°. *there exist $a, b \in \mathbb{R}$, $-\infty \leq a < 0 < b \leq +\infty$, such that*

$$f(x) = \begin{cases} x/a & x < a \\ 1 & a \leq x < b; \\ x/b & x \geq b \end{cases}$$

3°. *there exist* $a, b \in \mathbb{R}$, $0 < a \le b \le +\infty$, *such that*

$$f(x) = \begin{cases} 0 & x < 0 \\ f_{a,b}(x) & x \ge 0 \end{cases};$$

4°. *there exist* $a, b, c, d \in \mathbb{R}$, $0 < a \le b \le +\infty$, $0 < c \le d \le +\infty$, *such that*

$$f(x) = \begin{cases} f_{c,d}(-x) & x < 0 \\ f_{a,b}(x) & x \ge 0 \end{cases} \quad or \quad f(x) = \begin{cases} f_{c,d}(-x) & x < 0 \\ -f_{a,b}(x) & x \ge 0 \end{cases};$$

5°. $x \to f(-x)$, $x \in \mathbb{R}$, *where* f *is defined in* 2°, 3°, *or* 4°;
6°. $(-f)$ *where* f *is defined in* 2°, 3°, *or* 4°.

The proofs of Theorems 3 and 4 are based on Theorem 2.

Applying Theorem 1 one can get the following characterization of the sign function.

COROLLARY 2. *Let* $k \in \mathbb{N}$ *be fixed. Suppose that* $f: \mathbb{R} \to \mathbb{R}$ *is bounded on* (a, ∞) *for some* $a > 0$, *continuous on* $(-\infty, 0) \cup (0, \infty)$, *left and right discontinuous at* 0,

$$\liminf_{x \to 0+} f(x) > 0,$$

and there is an $x < 0$ *such that* $f(x) < 0$. *Then* f *is a solution of the functional equation*

$$f(x[f(x)]^{2k-1}) = (f(x))^{2k}, \qquad x \in \mathbb{R},$$

if, and only if,

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0. \\ 1 & x > 0 \end{cases}$$

Proof. Put $g := f|_{(0,\infty)}$ and suppose that there is a point $z > 0$ such that $g(z) = 0$. Since g is continuous, and positive in a right vicinity of 0, we may assume that $g(x) > 0$ for all $x \in (0, z)$. Take an arbitrary sequence $x_n \in (0, z)$ such that $\lim_{n \to \infty} x_n = z$, and put

$$y_n := x_n[f(x_n)]^{2k-1}, \qquad n \in \mathbb{N}.$$

Then of course $y_n > 0$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} y_n = 0$. From the functional equation we have

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} f(x_n[f(x_n)]^{2k-1}) = \lim_{n\to\infty} (f(x_n))^{2k} = 0,$$

which is a contradiction. thus $g(x) \neq 0$ for all $x > 0$ and, by the continuity of g, it is positive on $(0, \infty)$. It follows that

$$g(x[g(x)]^{2k-1}) = (g(x))^{2k}, \qquad x > 0.$$

As g is bounded and $\liminf_{x\to 0+} g(x)$ is positive, applying Theorem 1 we obtain $f(x) = g(x) = 1$ for all $x > 0$.

By assumption, the function f has negative values for some $x < 0$. We shall show that f takes only negative values on $(-\infty, 0)$. Indeed, otherwise we could find a $z < 0$ such that $f(z) = 0$, and a sequence $x_n < 0$ such that $\lim_{n\to\infty} x_n = z$, and $f(x_n) < 0$ for all $n \in \mathbb{N}$. Then

$$y_n := x_n[f(x_n)]^{2k-1} > 0, \qquad n \in \mathbb{N}.$$

Making use of what we have already shown,

$$(f(x_n))^{2k} = f(x_n[f(x_n)]^{2k-1}) = f(y_n) = 1, \qquad n \in \mathbb{N}.$$

It follows that $f(x_n) = -1$ for all $n \in \mathbb{N}$, and consequently

$$f(z) = \lim_{n\to\infty} f(x_n) = -1,$$

which is the desired contradiction. Thus we have proved that $f(x) < 0$ for all $x < 0$. Hence

$$(f(x))^{2k} = f(x[f(x)]^{2k-1}) = 1 \qquad \text{for all } x < 0,$$

and, consequently, $f(x) = -1$ for all $x < 0$. Since f is left and right discontinuous at 0, the relation $f(0) = 0$ follows from Remark 5. This completes the proof.

4. Discontinuous solutions

Denote by $\mathbb{Q}$ the set of rational numbers.

PROPOSITION. *Let* $p \in \mathbb{R}$, $p \neq 0$, *be fixed and suppose that a function* $f: (0, \infty) \to (0, \infty)$ *is rationally homogeneous, i.e.*

$$f(rx) = rf(x) \qquad r, x > 0, r \in \mathbb{Q}.$$

If the range of the function $(0, \infty) \ni x \to [f(x)]^p$ *is contained in* $\mathbb{Q}$, *then* f *is a solution of equation* (1).

The proof is obvious.

EXAMPLES. Let $F: [0, \infty)^2 \to [0, \infty)$ be an arbitrary homogeneous function. Then for two arbitrary additive functions $\alpha, \beta: \mathbb{R} \to \mathbb{R}$ with rational values only (cf. Kuczma [3], p. 120), the function $f: [0, \infty) \to [0, \infty)$ defined by

$$f(x) := F(|\alpha(x)|, |\beta(x)|), \qquad x \geq 0,$$

is rationally homogeneous. Taking here for instance

$$F(x, y) := x, \quad \text{or} \quad F(x, y) := \sqrt{x^2 + y^2}, \qquad x, y \geq 0,$$

in view of the above proposition, we get discontinuous functions f which satisfy equation (12) for every $p > 0$. Note that the graphs of these solutions are dense in $[0, \infty)^2$.

5. Final remarks

The functional equation (1) is a special case of

$$f(\mu x[f(x)]^p) = \lambda(f(x))^{p+1}, \tag{15}$$

where $\mu, \lambda > 0$ are fixed constants. This equation for $p = 1$ and $\mu = 2$ comes up in connection with some functional equations of several variables (cf. Aczél, Dhombres [1], p. 321). Note that equation (15), with three parameters p, μ, and λ, can be written in an equivalent form in which only two parameters occur. In fact, we have the following:

REMARK 9. Setting $g(x) := \lambda^{1/p} f(x)$, *and* $s := \mu^{-1}\lambda$, we can write equation (15) in the form

$$g(s^{-1}x[g(x)]^p) = (g(x))^{p+1}.$$

Now it is easy to see that $g \equiv 1$ is a solution of this equation. Consequently, the constant function $f \equiv \lambda^{-1/p}$ is a solution of the functional equation (15). Moreover we get the following

COROLLARY 3. *Let $p \in \mathbb{R}$, and $\mu > 0$ be fixed numbers. Then a function $f: (0, \infty) \to (0, \infty)$ is a solution of the functional equation*

$$f(\mu x[f(x)]^p) = \mu(f(x))^{p+1}, \tag{16}$$

if, and only if, the function $\mu^{1/p} f$ is a solution of equation (1).

Applying this corollary and Theorem 1, we can find the continuous solutions of equation (16).

REMARK 10. Let $p \in \mathbb{R}$, $p \neq 0$. Suppose that a continuous function $f: (0, \infty) \to (0, \infty)$ satisfies the functional equation

$$f(s^{-1}x[f(x)]^p) = (f(x))^{p+1}$$

with $s > 0$, $s \neq 1$. Setting

$$M(x) := s^{-1}x[f(x)]^p, \qquad x > 0; \qquad D(x) := \frac{f(x)}{x}, \qquad x > 0,$$

we can write this equation in the form

$$D(M(x)) = sD(x), \qquad x > 0.$$

A similar argument as in the proof of Theorem 1 shows that M is one-to-one and, consequently, strictly monotonic. Since $s \neq 1$, and $D \geq 0$, the function M is fixed point free. Thus M is strictly increasing and, consequently, either $M(x) < x$ for all $x > 0$, or $M(x) > x$ for all $x > 0$. Note also that, according to the definition of M, either $[f(x)]^p < s$, for all $x > 0$, or $[f(x)]^p > s$, for all $x > 0$. However the form of the general continuous solution for $s \neq 1$ is an open question.

REMARK 11. Let $c > 0$. Note that f is a solution of (16) if, and only if, the function $x \to f(cx)$ is a solution. All functional equations considered here have this property. Moreover, if the domain of an unknown function f is $\mathbb{R}$, this is also true for $c < 0$.

Acknowledgments

The authors would like to thank one of the referees whose remarks and suggestions allowed us to improve essentially the formulation of Theorem 1.

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