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## ENVELOPES OF SPECIAL CLASS OF ONE-PARAMETER FAMILIES OF CURVES

### Introduction

In our recent paper [6] concernig some relations between the logarithmic and arithmetic means we have obtained a family of solutions  $\phi_\alpha$  of a relevant one-parameter system of functional equations which are of the form

$$\phi_\alpha(x) = g\left(\frac{x}{\alpha}\right) + g(\alpha), \quad x > 0,$$

where  $\alpha > 0$  is the parameter, and  $g$  is a particular solution of the functional equation corresponding to the parameter  $\alpha = 1$ . Since  $g$  completely describes the one-parameter family of solutions  $(\phi_\alpha)_{\alpha > 0}$ , we call  $g$  to be its *generator*. Being interested in the mutual dependence of the position of graphs of the family  $\phi_\alpha$  on the parameter  $\alpha$ , it is natural to ask what is the envelope  $E_g$  of the family of curves

$$\mathcal{G}(g) := \{\text{graph}(\phi_\alpha) : \alpha > 0\}.$$

This question, in the context of the family  $\mathcal{G}(g)$ , appears to be interesting. There are some relationships between the envelope  $E_g$  of the logarithmic functions.

We also show that there are structural similarities between behaviour of the above mentioned class of curves  $\mathcal{G}(g)$  and its envelope  $E_g$ , and the classes of three families of curves being the graphs of the functions

$$\phi_\alpha(x) = g(x - \alpha) + g(\alpha), \quad x, \alpha \in \mathbb{R},$$

$$\phi_\alpha(x) = g(x - \alpha)g(\alpha), \quad x, \alpha \in \mathbb{R},$$

$$\phi_\alpha(x) = g\left(\frac{x}{\alpha}\right)g(\alpha), \quad x > 0,$$

and their envelopes, where  $g$  is appropriate defined. Some connections, respectively, with linear, exponential, and power functions, will be exhibited.

### 1. Envelopes for families $\mathcal{G}(g)$ of logarithmic type

By  $\mathbb{R}$  we denote the set of reals.

For an arbitrary function  $g : (0, \infty) \rightarrow \mathbb{R}$  define the one-parameter family of functions  $\phi_\alpha : (0, \infty) \rightarrow \mathbb{R}$  by

$$\phi_\alpha(x) := g\left(\frac{x}{\alpha}\right) + g(\alpha), \quad x, \alpha > 0,$$

of the generator  $g$ ; by  $\mathcal{G}(g)$  denote the family of curves being the graphs of  $\phi_\alpha$ ,  $\alpha > 0$ , and by  $E_g$ , the envelope of the family  $\mathcal{G}(g)$  (provided it exists).

We often identify a function and its graph. Therefore we write down the envelope  $E_g$  in the form  $y = E_g(x)$ ,  $x > 0$ , when it is possible and convenient.

**Remark. 1.1.** If  $g(x) = c \log x + g(1)$ ,  $x > 0$ , where  $c$  and  $g(1)$  are arbitrary real constant, then  $\mathcal{G}(g) = \{g\}$  is a singleton, and  $E_g$ , the envelope of  $g$ , obviously, coincides with the graph of  $g$ .

It turns out that the converse implication holds true:

**PROPOSITION 1.1.** *Let  $g : (0, \infty) \rightarrow \mathbb{R}$  be an arbitrary function. Then  $\mathcal{G}(g)$  is a singleton if, and only if, the function  $g$  satisfies the functional equation*

$$g(xy) + g(1) = g(x) + g(y), \quad x, y > 0.$$

*If moreover  $g$  is continuous at least at one point, then there exists a constant  $c \in \mathbb{R}$  such that  $g(x) = c \log x + g(1)$ ,  $x > 0$ .*

**Proof.** The family  $\mathcal{G}(g)$  is a singleton if, and only if,

$$g\left(\frac{x}{\alpha}\right) + g(\alpha) = g\left(\frac{x}{\beta}\right) + g(\beta)$$

for all  $x, \alpha, \beta > 0$ . Setting  $x = \beta$  gives

$$g(\beta/\alpha) + g(\alpha) = g(1) + g(\beta), \quad \alpha, \beta > 0.$$

Hence, for  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by the formula

$$\psi(\alpha) := g(\alpha) - g(1), \quad \alpha > 0,$$

one gets  $\psi(1) = 0$  and

$$(1) \quad \psi\left(\frac{\beta}{\alpha}\right) + \psi(\alpha) = \psi(\beta), \quad \alpha, \beta > 0.$$

Hence, setting  $\beta = 1$  in this equation we obtain

$$\psi\left(\frac{1}{\alpha}\right) = -\psi(\alpha), \quad \alpha > 0.$$

Now replacing  $\alpha$  by  $\alpha^{-1}$  in (1) gives

$$(2) \quad \psi(\alpha\beta) = \psi(\alpha) + \psi(\beta), \quad \alpha, \beta > 0,$$

which means that

$$g(\alpha\beta) + g(1) = g(\alpha) + g(\beta), \quad \alpha, \beta > 0.$$

The converse implication is obvious. Since  $\psi$  is a solution of the logarithmic Cauchy functional equation (2), the remaining statement is a well known fact (cf. for instance Aczél [1], p. 41). This completes the proof.

Remark 1.2. Note that the continuity of  $g$  at least at one point can be replaced by the measurability of  $g$ , or by the boundedness above (or below) in a neighbourhood of a point (cf. for instance Kuczma [4], p. 218).

The main result of this section reads as follows:

THEOREM 1.1. *Let  $g : (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function. Then the graph of the function*

$$(3) \quad (0, \infty) \ni x \rightarrow 2g(\sqrt{x}),$$

*is contained in the envelope of the family  $\mathcal{G}(g)$ . If the function*

$$(4) \quad (0, \infty) \ni x \rightarrow g'(x)x \text{ is one-to-one,}$$

*then the envelope  $E_g$  has the representation  $y = Eg(x) = 2g(\sqrt{x})$ ,  $x > 0$ .*

Proof. According to a classical method (cf. for instance Favard [2], Chapter III), to find the envelope of the family of curves  $\mathcal{G}(g)$  it is enough to eliminate the parameter  $\alpha$  from the system of equations

$$y = g(\alpha^{-1}x) + g(\alpha), \quad g'(\alpha^{-1}x)(-\alpha^{-2}x) + g'(\alpha) = 0, \quad x, \alpha > 0, y \in \mathbb{R}.$$

The second equation can be written in the following equivalent form

$$g'(\alpha^{-1}x)(\alpha^{-1}x) = g'(\alpha)\alpha, \quad x, \alpha > 0,$$

If the function  $(0, \infty) \ni x \rightarrow g'(x)x$  is one-to-one, it follows that  $\alpha^{-1}x = \alpha$ , and consequently,  $\alpha = \sqrt{x}$ ,  $x > 0$ . Setting  $\alpha = \sqrt{x}$  into the first of the equations we get the function

$$(5) \quad y = 2g(\sqrt{x}), \quad x > 0.$$

the graph of which is the envelope of the considered family of curves.

If the function  $(0, \infty) \ni x \rightarrow g'(x)x$  is not one-to-one, then, obviously, every point of the graph of the function (3), is a point of the envelope. This completes the proof.

Remark 1.3. If  $g(x) = c \log x + g(1)$ ,  $x > 0$ , then  $g'(x)x = c$  for all  $x > 0$ , i.e. the function (4) is constant. In particular it is not one-to-one. But of course (cf. Remark 1),  $E_g$  even coincides with the graph of  $g$ .

Remark 1.4. Denote by  $\mathcal{F}((0, \infty), \mathbb{R})$  the set of all functions  $\psi : (0, \infty) \rightarrow \mathbb{R}$ . For a given function  $F : \mathbb{R} \rightarrow \mathbb{R}$  define an operator  $T : \mathcal{F}((0, \infty), \mathbb{R}) \rightarrow \mathcal{F}((0, \infty), \mathbb{R})$  by the formula  $T(\psi) := F \circ \psi$ . Let  $\mathcal{G}(g)$  and  $\mathcal{G}(h)$  be the suitable families of curves of continuous generators  $g$  and  $h$ . Note that  $T(\mathcal{G}(g)) \subseteq$

$\mathcal{G}(h)$  if, and only if, there exists a function  $\beta : (0, \infty) \rightarrow (0, \infty)$  such that  $F, g$  and  $h$  satisfy the functional equation

$$F(g(x/\alpha) + g(\alpha)) = h(x/\beta(\alpha)) + h(\beta(\alpha)), \quad x, \alpha > 0.$$

Assuming that  $g : (0, \infty) \rightarrow \mathbb{R}$  is bijective and  $\beta(\alpha) = \alpha$  for all  $\alpha > 0$ , we shall prove that  $T(\mathcal{G}(g)) \subseteq \mathcal{G}(h)$  iff  $T$  is affine. In fact, as

$$F(g(x/\alpha) + g(\alpha)) = h(x/\alpha) + h(\alpha), \quad x, \alpha > 0,$$

setting  $x := \alpha^2$  gives  $F(2g(\alpha)) = 2h(\alpha)$  for all  $\alpha > 0$ . It follows that

$$F(x) = 2h \circ g^{-1}(x/2), \quad x \in \mathbb{R}.$$

Substituting this function into the previous relation we get

$$2h \circ g^{-1}\left(\frac{g(x/\alpha) + g(\alpha)}{2}\right) = h(x/\alpha) + h(\alpha), \quad x, \alpha > 0.$$

Replacing  $\alpha$  by  $y$ , and  $x$  by  $xy$ , gives

$$2h \circ g^{-1}\left(\frac{g(x) + g(y)}{2}\right) = h(x) + h(y), \quad x, y > 0.$$

Replacing here  $x$  by  $g^{-1}(x)$ , and  $y$  by  $g^{-1}(y)$ ,  $x, y \in \mathbb{R}$ , we obtain

$$h \circ g^{-1}\left(\frac{x + y}{2}\right) = \frac{h \circ g^{-1}(x) + h \circ g^{-1}(y)}{2}, \quad x, y \in \mathbb{R}.$$

It follows that there are  $a, b \in \mathbb{R}$  such that  $h \circ g^{-1}(x) = ax + b$  for all  $x \in \mathbb{R}$  (cf. for instance Aczél [1], p. 43), and consequently

$$F(x) = 2h \circ g^{-1}(x/2) = 2(ax/2 + b) = ax + 2b, \quad x \in \mathbb{R},$$

which was to be shown.

**Remark 1.5.** Note that an element  $\phi_\alpha$  of the family  $\mathcal{G}(g)$  coincides with the generator  $g$  if, and only if, there is an  $\alpha > 0$  such that  $g$  satisfies the functional equation

$$g(x) = \phi_\alpha(x) = g(x/\alpha) + g(\alpha), \quad x > 0.$$

In particular, if the generator  $g$  is strictly increasing and  $g(1) = 0$ , then  $\phi_1 = g$ .

**Remark 1.6.** In general, no member of the family  $\mathcal{G}(g)$  will coincide with the envelope  $E_g$ . If it is the case, then there exists an  $\alpha_0 > 0$  such that  $g$  satisfies the functional equation

$$2g(\sqrt{x}) = g(x/\alpha_0) + g(\alpha_0), \quad x > 0.$$

We shall prove that if  $g$  is differentiable at the point  $x = \alpha_0$ , and satisfies this equation, then there is  $c \in \mathbb{R}$  such that

$$(6) \quad g(x) = c \log x + g(\alpha_0), \quad x > 0.$$

Replacing  $x$  by  $\alpha_0^2 x$  we get

$$2g(\alpha_0\sqrt{x}) = g(\alpha_0 x) + g(\alpha_0), \quad x > 0.$$

Now it is easy to see that the function  $\phi : (0, \infty) \rightarrow \mathbb{R}$ ,

$$\phi(x) := g(\alpha_0 x) - g(\alpha_0), \quad x > 0,$$

satisfies the functional equation

$$2\phi(\sqrt{x}) = \phi(x), \quad x > 0,$$

and  $\phi$  is differentiable at the point  $x = 1$ . According to Fubini's result [3] (cf. also [5], p.394), there exists a constant  $c \in \mathbb{R}$  such that

$$\phi(x) = c \log x, \quad x > 0.$$

Hence we get the formula (6).

Thus, under the weak and natural assumption of the differentiability of the function  $g$ , the envelope  $E_g$  is a member of the family  $\mathcal{G}(g)$  if, and only if,  $\mathcal{G}(g)$  is a singleton with  $g = \log$ .

**Geometrical comments 1.1.** Let  $g : (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function, and suppose that the function  $x \rightarrow g'(x)x$  is one-to-one in  $\mathbb{R}_+$ . Then  $E_g$  coincides with the graph of the function  $x \rightarrow 2g(\sqrt{x})$ , so we can write  $E_g(x) = 2g(\sqrt{x})$ ,  $x > 0$ . Moreover, for every fixed  $\alpha > 0$ ,

$$y = g\left(\frac{x}{\alpha}\right) + g(\alpha), \quad x > 0,$$

the curve  $\phi_\alpha \in \mathcal{G}(g)$ , touches the envelope  $E_g$  at the point  $(\alpha^2, 2g(\alpha))$ . At this point of contact of the curves  $\phi_\alpha$  and  $E_g$ , the common tangent has the slope

$$\phi'(\alpha^2) = E'_g(\alpha^2) = g'(\alpha)/\alpha.$$

**EXAMPLE 1.** For the generator  $g(x) = x$ ,  $x > 0$ , we get the family  $\mathcal{G}(g)$  of functions  $\phi_\alpha$ ,

$$\phi_\alpha(x) = \frac{x}{\alpha} + \alpha, \quad x > 0.$$

Applying the above commentaries, we get the envelope  $E_g$ :

$$E_g(x) = 2\sqrt{x}, \quad x > 0,$$

points of contact:  $(\alpha^2, 2\alpha)$ ; slope of common tangent:  $1/\alpha$ .

## 2. Envelopes for families $\mathcal{G}(g)$ of affine type

Analogously as in the previous section, for an arbitrary generator func-

tion  $g : \mathbb{R} \rightarrow \mathbb{R}$  define the one-parameter family of functions  $\phi_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi_\alpha(x) := g(x - \alpha) + g(\alpha), \quad x, \alpha \in \mathbb{R},$$

and introduce the same notations:  $\mathcal{G}(g)$  and  $E_g$ .

**Remark 2.1.** If  $g(x) = cx + g(0)$ ,  $x \in \mathbb{R}$ , where  $c$  and  $g(0)$  are arbitrary real constants, then  $\mathcal{G}(g) = \{g\}$  is a singleton, and  $E_g$ , the envelope of  $g$ , coincides with the graph of  $g$ .

In an analogous way as Proposition 1.1 we can prove:

**PROPOSITION 2.1.** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function. Then  $\mathcal{G}(g)$  is a singleton if, and only if, the function  $g$  satisfies the functional equation*

$$g(x + y) + g(0) = g(x) + g(y), \quad x, y \in \mathbb{R}.$$

*If moreover  $g$  is continuous at least at one point, then there exists a constant  $c \in \mathbb{R}$  such that  $g(x) = cx + g(0)$ ,  $x \in \mathbb{R}$ .*

**THEOREM 2.1.** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function. Then the graph of the function  $\mathbb{R} \ni x \rightarrow 2g(\frac{x}{2})$ , is contained in the envelope of the family  $\mathcal{G}(g)$ . If the function  $\mathbb{R} \ni x \rightarrow g'(x)$  is one-to-one, then  $y = E_g(x) = 2g(\frac{x}{2})$ ,  $x \in \mathbb{R}$ .*

### 3. Envelopes for families $\mathcal{G}(g)$ of exponential type

Suppose that  $g : \mathbb{R} \rightarrow (0, \infty)$  is a generator of the one-parameter family of functions  $\phi_\alpha : \mathbb{R} \rightarrow (0, \infty)$ :

$$\phi_\alpha(x) := g(x - \alpha)g(\alpha), \quad x, \alpha \in \mathbb{R},$$

and let  $\mathcal{G}(g)$  and  $E_g$  be defined correspondingly.

**Remark 3.1.** If  $g(x) = g(0)e^{cx}$ ,  $x \in \mathbb{R}$ , where  $c \in \mathbb{R}$ , and  $g(0) > 0$ , are arbitrary constants, then  $\mathcal{G}(g) = \{g\}$  is a singleton, and  $E_g$ , the envelope of  $g$ , coincides with the graph of  $g$ .

**PROPOSITION 3.1.** *Let  $g : \mathbb{R} \rightarrow (0, \infty)$  be an arbitrary function. Then  $\mathcal{G}(g)$  is a singleton if, and only if, the function  $g$  satisfies the functional equation*

$$g(0)g(x + y) = g(x)g(y), \quad x, y \in \mathbb{R}.$$

*If moreover  $g$  is continuous at least at one point, then there exists a constant  $c \in \mathbb{R}$  such that  $g(x) = g(0)e^{cx}$ ,  $x \in \mathbb{R}$ .*

**THEOREM 3.1.** Let  $g : \mathbb{R} \rightarrow (0, \infty)$  be a differentiable function. Then the graph of the function

$$\mathbb{R} \ni x \rightarrow [g(\frac{x}{2})]^2$$

is contained in the envelope of the family  $\mathcal{G}(g)$ . If the function  $g'/g$  is one-to-one, then the envelope  $E_g$  has the representation

$$y = E_g(x) = [g(\frac{x}{2})]^2, \quad x > 0.$$

#### 4. Envelopes for families $\mathcal{G}(g)$ of power type

Suppose that  $g : (0, \infty) \rightarrow (0, \infty)$  is a generator of the one-parameter family of functions  $\phi_\alpha : (0, \infty) \rightarrow (0, \infty)$ :

$$\phi_\alpha(x) := g(\frac{x}{\alpha})g(\alpha), \quad x, \alpha > 0,$$

and let  $\mathcal{G}(g)$  and  $E_g$  be defined correspondingly.

**Remark 4.1.** If  $g(x) = g(1)x^c$ ,  $x > 0$ , where  $c \in \mathbb{R}$ , and  $g(1) > 0$ , are arbitrary constants, then  $\mathcal{G}(g) = \{g\}$  is a singleton, and  $E_g$ , the envelope of  $g$ , coincides with the graph of  $g$ .

**PROPOSITION 4.1.** Let  $g : (0, \infty) \rightarrow (0, \infty)$  be an arbitrary function. Then  $\mathcal{G}(g)$  is a singleton if, and only if, the function  $g$  satisfies the functional equation

$$g(1)g(xy) = g(x)g(y), \quad x, y > 0.$$

If moreover  $g$  is continuous at least at one point, then there exists a constant  $c \in \mathbb{R}$  such that  $g(x) = g(1)x^c$ ,  $x > 0$ .

**THEOREM 4.1.** Let  $g : (0, \infty) \rightarrow (0, \infty)$  be a differentiable function. Then the graph of the function

$$(0, \infty) \ni x \rightarrow [g(\sqrt{x})]^2$$

is contained in the envelope  $E_g$  of the family  $\mathcal{G}(g)$ . If the function

$$(0, \infty) \ni x \rightarrow \frac{g'(x)}{g(x)}x$$

is one-to-one, then the envelope curve has the representation

$$y = E_g(x) = [g(\sqrt{x})]^2, \quad x > 0.$$

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