Aequationes Mathematicae 43 (1992) 219–224
University of Waterloo

0001-9054/92/030219-06 $1.50 + 0.20/0

Functional inequality characterizing nonnegative concave functions in $(0, \infty)^k$

JANUSZ MATKOWSKI

Summary. In the present note we prove that every function $f: (0, \infty) \to [0, \infty)$ satisfying the inequality

$$af(s) + bf(t) \leqslant f(as + bt), \qquad s, t > 0,$$

for some a and b such that $0 < a < 1 < a + b$ must be of the form $f(t) = f(1)t$, $(t > 0)$. This improves our recent result in [2], where the inequality is assumed to hold for all $s, t \geqslant 0$, and gives a positive answer to the question raised there.

An analogue for functions of several real variables of the above result characterizes concave functions. Conjugate functions and some relations to Hölder's and Minkowski's inequalities are mentioned.

Introduction

In a recent paper [2] we have proved that, without any regularity conditions, *every function* $f: \mathbb{R}_+ \to \mathbb{R}_+$, $(\mathbb{R}_+ := [0, \infty))$, *satisfying the functional inequality*

$$af(s) + bf(t) \leqslant f(as + bt), \qquad (s, t \geqslant 0),$$

for some a, b *such that* $0 < a < 1 < a + b$ *must be of the form* $f(t) = f(1)t$, $(t \geqslant 0)$. It has also been shown that, using this result, one can get its analogue for functions of several real variables which, in turn, leads to a characterization of concave functions defined in $\mathbb{R}_+^k$, $(k \geqslant 2)$, to a new concept of conjugate function and to a simultaneous generalization of Hölder's and Minkowski's inequalities.

The long proof of this result heavily depended upon the assumption that 0 belongs to the domain of f. Nevertheless we conjectured that the theorem remains valid for every function $f: (0, \infty) \to \mathbb{R}_+$.

AMS (1980) subject classification (Revision 1985): Primary 39C05, 26B25, 26A51, Secondary 39B20.

Manuscript received December 17, 1990, and in final form, August 9, 1991.

In the present paper, making use of the above result as a lemma, we give a positive answer to the problem raised in [2]. This fact allows us to improve some other results presented there.

1. Two lemmas

We need the following

LEMMA 1. *Suppose that a and b are positive real numbers. If $f: (0, \infty) \to \mathbb{R}$ satisfies the inequality*

$$af(s) + bf(t) \leqslant f(as + bt), \qquad (s, t > 0),$$

then

$$\sum_{k=0}^{n} a^{n-k} b^k \sum_{i=1}^{\binom{n}{k}} f(s_{k,i}) \leqslant f\left(\sum_{k=0}^{n} a^{n-k} b^k \sum_{i=1}^{\binom{n}{k}} s_{k,i} \right)$$

for all $n \in \mathbb{N}$, $k = 0, \ldots, n$, $i = 1, \ldots, \binom{n}{k}$ and $s_{k,i} > 0$.

The proof is exactly the same as that of Step 1 in [2] so we omit it.

REMARK 1. In the proof of Step 1 in [2] it has been assumed that f is defined on $\mathbb{R}_+$. Note that, if $a + b = 1$, then $(0, \infty)$ in the above lemma can be replaced by an arbitrary interval. If $a \geqslant 1$ and $b \geqslant 1$ then instead of $(0, \infty)$ one can take (c, ∞) or $[c, \infty)$ where $c \geqslant 0$.

LEMMA 2. ([2], Theorem 1). *Let a and be be real numbers such that $0 < a < 1 < a + b$. If a function $f: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the inequality*

$$af(s) + bf(t) \leqslant f(as + bt), \qquad (s, t \geqslant 0),$$

then $f(t) = f(1)t$, $(t \geqslant 0)$.

2. Nonnegative solutions of the basic inequality

In this section we prove the following

THEOREM 1. *Let a and be be real numbers such that $0 < a < 1 < a + b$. If a function $f: (0, \infty) \to \mathbb{R}_+$ satisfies the inequality*

$$af(s) + bf(t) \leqslant f(as + bt), \qquad (s, t > 0), \tag{1}$$

then $f(t) = f(1)t$, $(t > 0)$.

Proof. Taking in (1) $s = t$ we have $(a+b)f(t) \leqslant f((a+b)t)$ and, by induction,

$$(a+b)^k f(t) \leqslant f((a+b)^k t), \qquad (t > 0; k \in \mathbb{N}).$$

Since $a + b > 1$ it follows that there exists a positive integer k such that $b_k := b(a+b)^k > 1$. Now we have from (1)

$$af(s) + b_k f(t) = af(s) + b(a+b)^k f(t) \leqslant af(s) + bf((a+b)^k t) \leqslant f(as + b_k t)$$

for all $s, t > 0$. Thus, without loss of generality, we can assume that $b > 1$.

It follows from Lemma 1 that

$$a^n b^m f(t) \leqslant f(a^n b^m t + s), \qquad (s, t > 0; n, m \in \mathbb{N}). \tag{2}$$

We shall use this inequality to show that f is nondecreasing. To this end consider the following two cases.

CASE 1. $\log b / \log a$ *is irrational.*

Since $0 < a < 1 < b$, the set

$$A := \{a^n b^m : n, m \in \mathbb{N}\}$$

is dense in $(0, \infty)$, (cf. [4], Lemma 4). Thus we can rewrite (2) as follows

$$\lambda f(t) \leqslant f(\lambda t + s), \qquad (\lambda \in A; s, t > 0).$$

Now take arbitrary $t, r > 0$. From the density of the set A it follows that there exists a sequence $\lambda_n \in A$, $(n \in \mathbb{N})$, such that

$$s_n := r + (1 - \lambda_n)t > 0; \qquad \lambda_n < 1, \quad (n \in \mathbb{N}),$$

and

$$\lim_{n \to \infty} \lambda_n = 1.$$

Setting in the above inequality $s := s_n$ and $\lambda := \lambda_n$ we obtain

$$\lambda_n f(t) \leqslant f(t + r)$$

for every $n \in \mathbb{N}$. Hence, letting $n \to \infty$, we get $f(t) \leqslant f(t + r)$ which shows that f is nondecreasing.

CASE 2. $\log b/\log a$ *is rational.*

Since $0 < a < 1 < b$ there exist positive integers n and m such that

$$\frac{\log b}{\log a} = -\frac{n}{m}, \qquad \text{i.e. that } a^n b^m = 1.$$

From (2) we get $f(t) \leqslant f(t+s)$ for all $s, t > 0$ which means that f is nondecreasing.

The monotonicity of f, which we just proved, implies that for every $t \geqslant 0$ there exists the right limit of f:

$$f(t+) := \lim_{r \to t+} f(r),$$

and, consequently, the function $g: \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula

$$g(t) := f(t+), \qquad (t \geqslant 0),$$

is well defined. Take now arbitrary nonnegative numbers s, t and two sequences $(s_n)_{n=1}^{\infty}, (t_n)_{n=1}^{\infty}$ such that

$$s < s_n, \quad t < t_n, \quad (n \in \mathbb{N}); \qquad \lim_{n\to\infty} s_n = s, \quad \lim_{n\to\infty} t_n = t.$$

We have $as + bt < as_n + bt_n$, $(n \in \mathbb{N})$, and, clearly,

$$af(s_n) + bf(t_n) \leqslant f(as_n + bt_n), \qquad (n \in \mathbb{N}).$$

Letting n tend to the ∞ we obtain

$$af(s+) + bf(t+) \leqslant f((as+bt)+), \qquad (s, t \geqslant 0),$$

or, equivalently,

$$ag(s) + bg(t) \leqslant g(as+bt), \qquad (s, t \geqslant 0).$$

By Lemma 2 we have $g(t) = g(1)t$, $(t \geqslant 0)$, i.e. $f(t+) = g(1)t$, $(t \geqslant 0)$. This and the monotonicity of f imply that $f(t) = f(1)t$, $(t > 0)$, which completes the proof.

REMARK 2. One of the referees of this paper noticed that in the above proof the full strength of Lemma 1 is not needed. It suffices to know (2). This can be explained as follows: For all $s, t > 0$,

$$af(t) \leqslant af(t) + bf(s/b) \leqslant f(at + b(s/b)) = f(at+s)$$

and hence $a^2 f(t) = a(af(t)) \leqslant af(at+s) \leqslant f(a(at+s)+u)$ for all $u > 0$. This implies that $a^2 f(t) \leqslant f(a^2 t + v)$ for all $t, v > 0$. By induction $a^n f(t) \leqslant f(a^n t + s)$ for all $s, t > 0$ and all $n \in \mathbb{N}$. Similarly $b^n f(t) \leqslant f(b^n t + s)$ for all $s, t > 0$ and all $m \in \mathbb{N}$. The last two inequalities imply (2).

3. A finite dimensional counterpart and conjugate functions

The following result is a counterpart of Theorem 1 for functions of several variables.

THEOREM 2. *Let* $k \in \mathbb{N}$, $k \geqslant 2$, *and suppose that* $0 < a < 1 < a + b$. *A function* $f\colon (0, \infty)^k \to \mathbb{R}_+$ *satisfies the inequality*

$$af(x) + bf(y) \leqslant f(ax + by), \qquad x, y \in (0, \infty)^k, \tag{3}$$

if and only if the following two conditions are satisfied:

(*i*) f *is positively homogeneous*;

(*ii*) *for each* $i = 1, \dots, k$, *the function* $f_i\colon (0, \infty)^{k-1} \to \mathbb{R}_+$ *given by the formula*

$$f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) := f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_k)$$

is concave in $(0, \infty)^{k-1}$.

Proof. Suppose that f satisfies (3). For an arbitrary fixed $x \in \mathbb{R}_+^k$ define a function $g\colon (0, \infty) \to \mathbb{R}_+$ by the formula

$$g(t) := f(tx), \qquad (t > 0).$$

From (3) we have

$$ag(s) + bg(t) = af(sx) + bf(tx) \leqslant f((as + bt)x) = g(as + bt)$$

for all $s, t > 0$. In view of Theorem 1 we get $g(t) = g(1)t$, $(t > 0)$, i.e. $f(tx) = tf(x)$, $(t > 0)$, which proves (i).

Hence we get

$$\begin{aligned} f(x_1, \dots, x_k) &= x_i f\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_k}{x_i}\right) \\ &= x_i f_i\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_k}{x_i}\right) \end{aligned} \tag{4}$$

for each $i = 1, \dots, k$ and all $x = (x_1, \dots, x_k) \in (0, \infty)^k$. Substituting this formula into (3) and then replacing x_j and y_j by x_j/a and y_j/b, respectively, we obtain

$$\begin{aligned} &x_i f_i\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \dots, \frac{x_k}{x_i}\right) + y_i f_i\left(\frac{y_1}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_k}{y_i}\right) \\ &\leqslant (x_i + y_i) f_i\left(\frac{x_1 + y_1}{x_i + y_i}, \dots, \frac{x_{i-1} + y_{i-1}}{x_i + y_i}, \frac{x_{i+1} + y_{i+1}}{x_i + y_i}, \dots, \frac{x_k + y_k}{x_i + y_i}\right) \end{aligned} \tag{5}$$

for all $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in (0, \infty)^k$ and $i = 1, \ldots, k$. It is easy to verify that this inequality is equivalent to the concavity of f_i, which proves (ii).

Conversely, if a function $f: (0, \infty)^k \to \mathbb{R}_+$ satisfies conditions (i) and (ii) then relations (4) and (5) hold true. Replacing in (5) x_j and y_j respectively by ax_j and by_j, $(j = 1, \ldots, k)$, and making use of (4) we get (3). This completes the proof.

REMARK 3. This result improves Theorem 3 in [2] where inequality (3) is assumed to hold for all $x, y \in \mathbb{R}_+^k$.

REMARK 4. The functions $f_1, \ldots, f_k$ mentioned in Theorem 2(i) are closely interrelated. In fact, for $i, j \in \{1, \ldots, k\}$, $i \neq j$, we have

$$x_i f_i\left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_k}{x_i}\right) = x_j f_j\left(\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_k}{x_j}\right).$$

In this connection we introduced a notion of conjugate functions (cf. [2], [3], [5]). It generalizes the power conjugate functions. Moreover, inequality (5), satisfied by each of the functions $f_1, \ldots, f_k$, generalizes Hölder's and Minkowski's inequalities. Let us mention that the conjugate functions appear also in connection with some mean value theorems (cf. J. Aczél [1]).

REFERENCES

[1] ACZÉL, J., *Remark No. 15.* In: *Report of Meeting. The 27th International Symposium on Functional Equations.* Aequationes Math. *39* (1980), 308.

[2] MATKOWSKI, J., *Functional inequality characterizing convex functions, conjugacy, and a generalization of Hölder's and Minkowski's inequalities.* Aequationes Math. *40* (1990), 168–180.

[3] MATKOWSKI, J., *On a generalization of Hölder's and Minkowski's inequalities and symmetric norms in* $\mathbb{R}^n$. In: *Report of Meeting. The 27th International Symposium on Functional Equations.* Aequationes Math. *39* (1990), 287–288.

[4] MATKOWSKI, J., *The converse of the Minkowski's inequality theorem and its generalization.* Proc. Amer. Math. Soc. *109* (1990), 663–675.

[5] MATKOWSKI, J., *A generalization of Hölder's and Minkowski's inequalities and conjugate functions, Constantin Carthéodory: An International Tribute*, edited by Th. M. Rassias. World Scientific Publ. Co., Singapore, 1991, pp. 819–827.

Department of Mathematics,
Technical University,
Willowa 2,
PL-43-309 Bielsko-Biala,
Poland.