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ABSTRACT. Let C be a cone in a linear space. Under some weak regularity conditions we show that every subadditive function $\mathbf{p}: C \to R$ such that $\mathbf{p}(rx) \leqslant r\mathbf{p}(x)$ for some $r \in (0, 1)$ and all $x \in C$ must be positively homogenous. As an application we obtain a new characterization of L^p-norm. This permits us to prove among other things the following converse of Minkowski's inequality.

Let (Ω, Σ, μ) be a measure space such that there exist disjoint sets A, $B \in \Sigma$ satisfying the condition $\mu(B) = 1/\mu(A)$, $\mu(A) \neq 1$. If $\varphi: R_+ \to R_+$ is an arbitrary bijection such that

$$\varphi^{-1}\left(\int_\Omega \varphi \circ (x + y) d\mu \right) \leqslant \varphi^{-1}\left(\int_\Omega \varphi \circ x d\mu \right) + \varphi^{-1}\left(\int_\Omega \varphi \circ y d\mu \right)$$

for all the μ-integrable step functions x, $y: \Omega \to R_+$ then φ is a power function.

Introduction

Let R, R_+ and N denote respectively the set of reals, nonnegative reals and positive integers.

For a measure space (Ω, Σ, μ) let $\mathbf{S} = \mathbf{S}(\Omega, \Sigma, \mu)$ stand for the linear space of all the μ-integrable step functions $x: \Omega \to R$ and let $\mathbf{S}_+ := \{x \in S: x \geqslant 0\}$.

It can be easily verified that for every bijection $\varphi: R_+ \to R_+$ such that $\varphi(0) = 0$ the functional $P_\varphi: \mathbf{S} \to R_+$ given by the formula

$$(1) \qquad P_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x| d\mu \right), \qquad x \in \mathbf{S},$$

is well defined. In [4] we have proved the following *converse of Minkowski's inequality.*

Let (Ω, Σ, μ) be a measure space with two sets A, $B \in \Sigma$ such that

$$(2) \qquad 0 < \mu(A) < 1 < \mu(B) < \infty$$

and $\varphi \colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ a bijection such that $\varphi(0) = 0$. *If φ^{-1} is continuous at 0 and*

$$(3) \qquad P_\varphi(x + y) \leqslant P_\varphi(x) + P_\varphi(y), \qquad x, \ y \in S_+,$$

then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

It has also been shown that condition (2) is essential. In this paper we show that modifying the definition of P_φ one can eliminate the assumption $\varphi(0) = 0$. The remaining assumption of the continuity of φ^{-1} at 0 plays a key but technical role. We conjecture that the above result is valid without this assumption. However it seems to be a difficult problem to get rid of it completely.

In a recent paper [7] we have attempted to replace the continuity of φ^{-1} at 0 by the following assumption: there exist disjoint sets C, $D \in \Sigma$ of positive measures such that $\mu(C) + \mu(D) = 1$. This approach leads to some open problems in the theory of convex functions. Nevertheless we were able to prove that in the case when $\mu(C) = \mu(D)$ the continuity of φ^{-1} at 0 is superfluous.

In section 3 of the present paper we show that the continuity of φ^{-1} at 0 together with assumption (2) can be replaced by one of the following conditions:

(i) *there exist $n \in N$, $n > 1$, and A, $B \in \Sigma$ such that*

$$A \cap B = \varnothing; \qquad \mu(A) = \frac{1}{n}; \qquad \mu(B) = n,$$

or

(ii) *there exist n, $m \in N$, $n \neq m$, $n > 1$, and A, B, $C \in \Sigma$ such that*

$$A \cap B = \varnothing; \qquad \mu(A) = \frac{m}{n}; \qquad \mu(B) = \frac{n}{m}; \qquad \mu(C) = n.$$

The proof of this theorem is based on the following characterization of L^p-norm which is the main result of section 2.

If (Ω, Σ, μ) is a measure space with two disjoint sets A, $B \in \Sigma$ such that $\mu(A) = \mu(B) = 1$; a function $\varphi \colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is bijective, inequality (3) holds and there exists an $r \in (0, 1)$ such that $P_\varphi(rx) \leqslant r P_\varphi(x)$ for all $x \in S_+$ then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.

This is a partial generalization of a theorem in [5] where P_φ is supposed to be positively homogeneous. A keystone of the proof is a recently obtained theorem which roughly speaking states that (under some weak regularity conditions) every real subadditive function **p** defined on a cone **C** in a linear space satisfying condition that there exists an $r \in (0, 1)$ such that $\mathbf{p}(rx) \leqslant r\mathbf{p}(x)$ for every $x \in \mathbf{C}$ must be positively homogeneous (cf. [8] and [9]). In the preparatory section 1 we give a sketch of the proof of this result.

1. Auxiliary results

Let X be a real linear space. A set $C \subset X$ is said to be a cone in X if $C + C \subset C$ and $tC \subset C$ for every $t > 0$.

LEMMA 1. *Let* X *be a real linear space and* C *a cone in* X. *If* $p: C \to R$ *satisfies the following conditions:*

1°. p *is subadditive i.e.* $p(x + y) \leqslant p(x) + p(y)$ *for all* $x, y \in C$;

2°. *for every* $x \in C$ *the function* $f_x: (0, \infty) \to R$ *given by the formula*

$$f_x(t) := p(tx), \qquad t > 0,$$

is bounded above in a neighbourhood of a point;

3°. *there exists an* $r \in (0, 1)$ *such that*

$$p(rx) \leqslant rp(x), \qquad x \in C,$$

then p *is positively homogeneous i.e.* $p(tx) = tp(x)$ *for all* $t > 0$, $x \in C$.

PROOF. (Sketch) Take an arbitrary $x \in C$. By 1° the function $f := f_x$ is subadditive in $(0, \infty)$. This together with 2° implies that f is locally bounded above, (i.e. bounded above on every compact subset of $(0, \infty)$), and, consequently, locally bounded. Therefore (cf. [2], Theorem 7.6.1, p. 244 and the remark coming after its proof; also [3], p. 407)

$$\lim_{t \to \infty} \frac{f(t)}{t} = \inf_{t > 0} \frac{f(t)}{t}. \tag{4}$$

By induction from 3° we have

$$\frac{f(t)}{t} \leqslant \frac{f(r^{-n}t)}{r^{-n}t}, \qquad t > 0; \quad n \in N.$$

Letting $n \to \infty$ and making use of (4) we hence obtain for all $t > 0$

$$\frac{f(t)}{t} \leqslant \inf_{t > 0} \frac{f(t)}{t},$$

which means that $f(t) = f(1)t$ for all $t > 0$. Now by the definition of f we have

$$p(tx) = f_x(t) = f(t) = f(1)t = f_x(1)t = p(x)t$$

which was to be shown.

REMARK 1. The same argument permits us to get more general result. Namely, instead of 1° we can assume that for every $x \in C$ the function f_x is subadditive in $(0, \infty)$ and instead of 2° that for every $x \in C$ there is $r_x \in (0, 1)$

such that every $t > 0$ we have $f_x(r_x tx) \leqslant r_x f_x(tx)$, (cf. [8] where a detailed proof is given).

We quote the following result due to T. Świątkowski and the present author (cf. [6]).

LEMMA 2. *Let* $f: \mathbf{R}_+ \to \mathbf{R}_+$ *be a subadditive bijection. If* f *is continuous at* 0 *then it is a homeomorphism of* $\mathbf{R}_+$.

REMARK 2. Let $x \in \mathbf{S}$. Then there exist disjoint $A_1, \ldots, A_k \in \Sigma$ and $x_1, \ldots, x_k \in \mathbf{R}$ such that

$$x = \sum_{i=1}^{k} x_i \chi_{A_i}; \qquad \mu(A_i) < \infty, \qquad (i = 1, \ldots, k).$$

(χ_E denotes the characteristic function of a set E). For an arbitrary bijection $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ we have

$$\varphi \circ |x| = \sum_{i=1}^{k} \varphi(|x_i|) \chi_{A_i} + \varphi(0) \chi_{\Omega - A_i}.$$

If $\varphi(0) = 0$ then $x \in \mathbf{S} \Rightarrow \varphi \circ |x| \in \mathbf{S}$, and, consequently, the functional P_φ is well defined for a given measure space (Ω, Σ, μ).

It is easily seen that in the case when $\mu(\Omega) < \infty$ the functional P_φ is well defined by the formula (1) even when the condition $\varphi(0) = 0$ fails to hold. One can also avoid this assumption in the case $\mu(\Omega) = \infty$ modifying the formula (1) as follows

$$P_\varphi(x) := \varphi^{-1}\left(\int_{\Omega_x} \varphi \circ |x| \, d\mu \right), \qquad x \in \mathbf{S},$$

where $\Omega_x := \{\omega \in \Omega : x(\omega) \neq 0\}$. Thus the assumption $\varphi(0) = 0$ in [4] was made to simplify the notations. From the next lemma it follows that it could be done without any loss of generality.

LEMMA 3. *Let* (Ω, Σ, μ) *be a measure space with at least one set* $A \in \Sigma$ *of positive finite measure such that* $\mu(A) \neq 1$ *and* $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ *an arbitrary bijection satisfying inequality* (3). *Then* $\varphi(0) = 0$.

PROOF. Let $a := \mu(A)$. Putting in (3) $x = y := t\chi_A$, $t \geqslant 0$, we obtain

$$\varphi^{-1}(a\varphi(2t)) \leqslant 2\varphi^{-1}(a\varphi(t)), \qquad t \geqslant 0,$$

which means that the function $f := \varphi^{-1} \circ (a\varphi)$ satisfies the inequality

$$f(2t) \leqslant 2f(t), \qquad t \geqslant 0.$$

Since f is a bijection of $\mathbf{R}_+$ there is a $t_0 \in \mathbf{R}_+$ such that $f(t_0) = 0$. From

the above inequality we infer that $f(2t_0) = 0$ and, consequently, $f(2t_0) = f(t_0)$. Now the bijectivity of f implies that $t_0 = 0$. Hence we get $\varphi^{-1}(a\varphi(0)) = 0$ and, since $a \neq 1$, $\varphi(0) = 0$. This completes the proof.

2. A characterization of L^p-norm

In this section we prove the following

THEOREM 1. *Let (Ω, Σ, μ) be a measure space with at least two sets A, $B \in \Sigma$ such that*

(5) $$A \cap B = \varnothing, \quad \mu(A) = \mu(B) = 1,$$

and suppose that $\varphi \colon R_+ \to R_+$ is bijective. If

(6) $$P_\varphi(x + y) \leqslant P_\varphi(x) + P_\varphi(y), \quad x, \ y \in \mathbf{S}_+,$$

and there exists an $r \in (0, 1)$ such that for every $x \in \mathbf{S}_+$

(7) $$P_\varphi(rx) \leqslant r P_\varphi(x),$$

then $\varphi(t) = \varphi(1) t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.

PROOF. To apply Lemma 1 put $\mathbf{X} := \mathbf{R}^2$, $\mathbf{C} := \mathbf{R}^2_+$ and define $\mathbf{p} \colon \mathbf{C} \to \mathbf{R}$ by

$$\mathbf{p}(x) := P_\varphi(x_1 \chi_A + x_2 \chi_B), \quad x = (x_1, x_2) \in \mathbf{R}^2_+.$$

From (6) and (7) the assumptions $1°$ and $3°$ of Lemma 1 are satisfied. To verify that condition $2°$ of this lemma is also fulfilled, we note that by the definitions of $\mathbf{p}$ and P_φ and (5) we get

(8) $$\mathbf{p}(x) = \varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \quad x = (x_1, x_2) \in \mathbf{R}^2_+.$$

As $\mathbf{p}$ is subadditive in $\mathbf{C}$ we have

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all nonnegative x_1, x_2, y_1, y_2. Since $\mu(A \cup B) = 2$ it follows from Lemma 3 that $\varphi(0) = 0$. Therefore substituting $y_1 = x_2 := 0$ we obtain $\varphi^{-1}(\varphi(x_1) + \varphi(y_2)) \leqslant x_1 + y_2$ or, equivalently,

(9) $$\mathbf{p}(x) = \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) \leqslant x_1 + x_2, \quad x_1, \ x_2 \geqslant 0.$$

Hence $f_x(t) := \mathbf{p}(tx) = \varphi^{-1}(\varphi(tx_1) + \varphi(tx_2)) \leqslant t(x_1 + x_2)$ which shows that condition $2°$ of Lemma 1 is fulfilled. According to this lemma we have $\mathbf{p}(tx) = t\mathbf{p}(x)$ for all $x \in \mathbf{C}$ and $t > 0$ which, in view of (8), can be written as

$$\varphi^{-1}(\varphi(tx_1) + \varphi(tx_2)) = t\varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \quad x_1, \ x_2 \geqslant 0; \ t > 0.$$

Replacing here x_1 by $\varphi^{-1}(x_1)$ and x_2 by $\varphi^{-1}(x_2)$ and making use of the bijectivity of φ we obtain

$$\varphi(t\varphi^{-1}(x_1 + x_2)) = \varphi(t\varphi^{-1}(x_1)) + \varphi(t\varphi^{-1}(x_2)), \qquad x_1, x_2 \geqslant 0, \quad t > 0,$$

which means that for every $t > 0$ the function $\varphi \circ (t\varphi^{-1})$ is additive. Since $\varphi \circ (t\varphi^{-1})$ is nonnegative, it must be a linear function (cf. J. Aczél [1], p. 34). Consequently, for every $t > 0$, there exists an $m(t) > 0$ such that

$$(10) \qquad \varphi(t\varphi^{-1}(x)) = m(t)x, \qquad x \geqslant 0.$$

Note that this relation remains valid if we additionally define $m(0) := 0$. Take arbitrary s, $t \geqslant 0$. Composing the functions $\varphi \circ (s\varphi^{-1})$ and $\varphi \circ (t\varphi^{-1})$ and making use of relation (10) we get

$$\varphi(st\varphi^{-1}(x)) = m(s)m(t)x, \qquad x \geqslant 0.$$

On the other hand the same relation says that

$$\varphi(st\varphi^{-1}(x)) = m(st)x, \qquad x \geqslant 0.$$

Hence we infer that

$$m(st) = m(s)m(t), \qquad s, \ t \geqslant 0,$$

i.e. m is multiplicative, and, in view of (10), m is bijective and

$$\varphi^{-1}(t) = \varphi^{-1}(1)m^{-1}(t), \qquad t \geqslant 0.$$

Now from (8) and from the multiplicativity of m and m^{-1} we have

$$\mathbf{p}(x) = m^{-1}(m(x_1) + m(x_2)), \qquad x = (x_1, x_2) \in R_+^2,$$

and, as $\mathbf{p}$ is subadditive,

$$(11)$$
$$m^{-1}(m(x_1 + y_1) + m(x_2 + y_2)) \leqslant m^{-1}(m(x_1) + m(x_2)) + m^{-1}(m(y_1) + m(y_2))$$

for all $x_1, x_2, y_1, y_2 \geqslant 0$. Setting here $x_1 = y_1 := s$ and $x_2 = y_2 := t$, we get

$$m^{-1}(2m(s + t)) \leqslant 2m^{-1}(m(s) + m(t)), \qquad s, \ t \geqslant 0.$$

From the multiplicativity of m^{-1} we obtain

$$m^{-1}(2)(s + t) \leqslant 2m^{-1}(m(s) + m(t)), \qquad s, \ t \geqslant 0.$$

This implies that for s, $t \geqslant 0$ and $c := m^{-1}(2)/2$ we have

$$cm^{-1}(t) \leqslant m^{-1}(s + t), \qquad c > 0,$$

and, consequently,

$$c \cdot \lim \sup_{t \to 0} m^{-1}(t) \leqslant \inf \{m^{-1}(s): s > 0\}.$$

Since m is bijective it follows that

$$\lim_{t \to 0} m^{-1}(t) = 0 = m^{-1}(0)$$

i.e. the function m^{-1} is continuous at 0. Setting in (11): $x_1 := m^{-1}(s)$, $y_2 := m^{-1}(t)$, $x_2 = y_1 := 0$ we get

$$m^{-1}(s + t) \leqslant m^{-1}(s) + m^{-1}(t), \qquad s, \ t \geqslant 0,$$

i.e. m^{-1} is subadditive in R_+. By Lemma 2, m^{-1} is a homeomorphism of R_+. Consequently (cf. J. Aczél [1], p. 41), there is a $p > 0$ such that $m(t) = t^p$ for all $t \geqslant 0$. Hence $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, which completes the proof.

REMARK 3. It is quite obvious that condition (7) of Theorem 1 is fulfilled if there exists an $r > 1$ such that for every $x \in S_+$:

$$P_\varphi(rx) \geqslant rP_\varphi(x).$$

Moreover, according to Remark 1, both these conditions can be replaced by more general ones.

Taking in Theorem 1 the measure space (Ω, Σ, μ) such that $\Omega := \{1, 2\}$; $\Sigma := 2^\Omega$; $\mu(\{1\}) = \mu(\{2\}) := 1$ and making use of Remark 3 we obtain the following

COROLLARY 1. Let $\varphi: R_+ \to R_+$ be a bijection such that

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all nonnegative x_1, y_1, x_2, y_2. If there exists an $r \in (0, 1)$, (resp. $r > 1$), such that

$$\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) \leqslant r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \qquad x_1, \ x_2 \geqslant 0,$$

(resp. the reversed inequality holds), then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.

REMARK 4. If a bijection $\varphi: R_+ \to R_+$ satisfies the functional equation

$$\varphi(rt) = \rho\varphi(t), \qquad t > 0,$$

for some positive r and ρ, $r \neq \rho$, then

$$\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) = r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \qquad x_1, \ x_2 \geqslant 0.$$

Indeed, we have $\varphi^{-1}(\rho t) = r\varphi^{-1}(t)$, $(t > 0)$, and, therefore

$$\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) = \varphi^{-1}(\rho[\varphi(x_1) + \varphi(x_2)])$$
$$= r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)).$$

3. A converse of Minkowski's inequality

In the previous section we have proved that if the functional P_φ satisfies the triangle inequality and a kind of substitute of the homogeneity condition, (cf. e.g. (7)), then φ must be a power function. Now we assume that P_φ satisfies only the triangle inequality.

The main result of this section reads as follows.

THEOREM 2. *Let* (Ω, Σ, μ) *be a measure space such that there exist* A, B, $C \in \Sigma$ *and* m, $n \in N$, $m \neq n$, *satisfying the following conditions:*

$$A \cap B = \varnothing; \qquad \mu(A) = \frac{m}{n}; \qquad \mu(B) = \frac{n}{m}; \qquad \mu(C) = n.$$

If $\varphi: R_+ \to R_+$ *is a bijection such that*

$$P_\varphi(x + y) \leqslant P_\varphi(x) + P_\varphi(y), \qquad x, y \in S_+,$$

then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

PROOF. By Lemma 3 we have $\varphi(0) = 0$. Hence, substituting in the assumed triangle inequality

$$x := \varphi^{-1}\left(\frac{s}{\mu(A)}\right)\chi_A, \qquad y := \varphi^{-1}\left(\frac{t}{\mu(B)}\right)\chi_B,$$

we get

$$\varphi^{-1}(s + t) \leqslant \varphi^{-1}(s) + \varphi^{-1}(t), \qquad s, t \geqslant 0,$$

i.e. φ^{-1} is subadditive. By induction we have for every $k \in N$

$$\varphi^{-1}(t_1 + \cdots + t_k) \leqslant \varphi^{-1}(t_1) + \cdots + \varphi^{-1}(t_k), \qquad t_1, \ldots, t_k \geqslant 0.$$

Setting here $t_1 = \cdots = t_k = t$ we get $\varphi^{-1}(kt) \leqslant k\varphi^{-1}(t)$ and, consequently,

$$\varphi^{-1}(k\varphi(t)) \leqslant kt, \qquad k \in N, \quad t \geqslant 0.$$

This implies that for every $k \in N$ the function $\varphi^{-1} \circ (k\varphi)$ is continuous at 0. Substituting in the triangle inequality in turn

$$x := s\chi_A, \quad y := t\chi_A; \qquad x := s\chi_B, \quad y := t\chi_B; \qquad x := s\chi_C, \quad y := t\chi_C,$$

we infer that the functions $\varphi^{-1} \circ \left(\dfrac{m}{n}\varphi\right)$, $\varphi^{-1} \circ \left(\dfrac{n}{m}\varphi\right)$ and $\varphi^{-1} \circ (n\varphi)$ are subadditive in R_+. From Lemma 2 it follows that $\varphi^{-1} \circ (n\varphi)$ is a homeomorphism of R_+. Since the composition of an increasing subadditive function and subadditive one is subadditive, the relation

$$\varphi^{-1} \circ (m\varphi) = (\varphi^{-1} \circ (n\varphi)) \circ \left(\varphi^{-1} \circ \left(\frac{m}{n} \varphi \right) \right)$$

implies that $\varphi^{-1} \circ (m\varphi)$ is subadditive and, by Lemma 2, a homeomorphism of R_+. The function $\varphi^{-1} \circ \left(\frac{1}{n} \varphi \right)$ being the inverse of $\varphi^{-1} \circ (n\varphi)$ is a homeomorphism of R_+. Now the relation

$$\varphi^{-1} \circ \left(\frac{m}{n} \varphi \right) = (\varphi^{-1} \circ (m\varphi)) \circ \left(\varphi^{-1} \circ \left(\frac{1}{n} \varphi \right) \right)$$

implies that $\varphi^{-1} \circ \left(\frac{m}{n} \varphi \right)$ and its inverse $\varphi^{-1} \circ \left(\frac{n}{m} \varphi \right)$ are homeomorphisms. Because these functions are inverses of one another and subadditive, they must be superadditive and, consequently, additive. Therefore (cf. J. Aczél [1], p. 34) there exists an $r > 0$ such that

$$\varphi^{-1} \left(\frac{m}{n} \varphi(t) \right) = rt, \qquad t \geqslant 0.$$

Denoting $a := \mu(A) = \frac{m}{n}$, we hence get

$$(12) \qquad a\varphi(t) = \varphi(rt), \qquad \varphi^{-1}(at) = r\varphi^{-1}(t), \qquad t \geqslant 0.$$

Setting in the triangle inequality

$$x := x_1 \chi_A + x_2 \chi_B, \qquad y := y_1 \chi_A + y_2 \chi_B; \qquad x_1, x_2, y_1, y_2 \geqslant 0,$$

and taking into account that $A \cap B = \varnothing$ and $\mu(B) = \frac{1}{a}$ we obtain

$$\varphi^{-1} \left(a\varphi(x_1 + y_1) + \frac{1}{a} \varphi(x_2 + y_2) \right) \leqslant \varphi^{-1} \left(a\varphi(x_1) + \frac{1}{a} \varphi(x_2) \right)$$
$$+ \varphi^{-1} \left(a\varphi(y_1) + \frac{1}{a} \varphi(y_2) \right).$$

Applying (12) we can write this inequality as follows

$$\varphi^{-1} \left(\varphi(rx_1 + ry_1) + \varphi\left(\frac{1}{r} x_2 + \frac{1}{r} y_2 \right) \right) \leqslant \varphi^{-1} \left(\varphi(rx_1) + \varphi\left(\frac{1}{r} x_2 \right) \right)$$
$$+ \varphi^{-1} \left(\varphi(ry_1) + \varphi\left(\frac{1}{r} y_2 \right) \right).$$

Replacing rx_1, $r^{-1}x_2$, ry_1, $r^{-1}y_2$ resp. by x_1, x_2, y_1, y_2 we get

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all nonnegative x_1, x_2, y_1, y_2. Applying once more (12) we obtain

$$\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) = \varphi^{-1}(a\varphi(x_1) + a\varphi(x_2))$$
$$= \varphi^{-1}(a[\varphi(x_1) + \varphi(x_2)])$$
$$= r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)).$$

Now our theorem results from Corollary 1 because, clearly, $r \neq 1$.

If in the above theorem $n = 1$ we can take $C = B$. Therefore we have the following

COROLLARY 2. *Let* (Ω, Σ, μ) *be a measure space such that there exist* A, $B \in \Sigma$ *and* $m \in N$, $m \neq 1$, *satisfying the following conditions:*

$$A \cap B = \varnothing; \qquad \mu(A) = m; \qquad \mu(B) = \frac{1}{m}.$$

If $\varphi: R_+ \to R_+$ *is a bijection such that*

$$P_\varphi(x + y) \leqslant P_\varphi(x) + P_\varphi(y), \qquad x, \ y \in S_+,$$

then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

Finally let us note that using Lemma 3 we can write the converse of Minkowski's inequality quoted in the introduction in a little more general form (cf. [4]).

THEOREM 3. *Let* (Ω, Σ, μ) *be a measure space with at least two sets* A, $B \in \Sigma$ *such that* $0 < \mu(A) < 1 < \mu(B) < \infty$. *If* $\varphi: R_+ \to R_+$ *is a bijection such that* φ^{-1} *is continuous at* 0 *and*

$$P_\varphi(x + y) \leqslant P_\varphi(x) + P_\varphi(y), \qquad x, \ y \in S_+,$$

then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

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