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Positive Homogeneous Functionals Related to $\mathbf{L}^p$-Norms

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One of the results reads as follows. Let $(\Omega\ \Sigma, \mu)$ be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $m, \phi, \psi: (0, \infty) \to (0, \infty)$ are functions such that ϕ and ψ are bijective, and $\phi(1) = 1 = \psi(1)$. Then

$$\psi\left(\int_{\Omega(X)} \phi \circ |tx|\, d\mu\right) = m(t)\psi\left(\int_{\Omega(X)} \phi \circ |x|\, d\mu\right) \qquad (*)$$

for all nonnegative simple functions $x: \Omega \to \mathbf{R}$, $x \not\equiv 0$ μ-a.e., and all $t > 0$, where $\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}$, if, and only if, m, ϕ, ψ are multiplicative and $\psi = m \circ \phi^{-1}$. If, moreover, arbitrary two functions chosen from the set $\{m, \phi, \psi\}$ satisfy some modest regularity assumptions then the homogeneity relation $(*)$ holds true if, and only if, m, ϕ, and ψ are the power functions.

INTRODUCTION

For a measure space (Ω, Σ, μ) denote by $\mathbf{S} = \mathbf{S}(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable simple functions $x: \Omega \to \mathbf{R}$, and by $\mathbf{S}_+ = \mathbf{S}_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $x \in \mathbf{S}$. It is easy to see that for arbitrary functions $\phi, \psi: (0, \infty) \to (0, \infty)$ the functional $\mathbf{p}_{\phi, \psi}: \mathbf{S} \to [0, \infty)$ given by

$$p_{\phi,\psi}(x) := \begin{cases} \psi\left(\int_{\Omega(x)} \phi \circ |x|\, d\mu\right) & \text{if } \mu(\Omega(x)) > 0 \\ 0 & \text{if } \mu(\Omega(x)) = 0 \end{cases}, \qquad x \in \mathbf{S},$$

where $\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}$, is well defined [5, Remark 5].

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Note that for $\phi(t) := \phi(1)t^p$, $t > 0$, where $p \in \mathbf{R} \setminus \{0\}$ is arbitrary fixed, and $\psi = \phi^{-1}$ (the inverse of ϕ), we have

$$p_{\phi,\psi}(x) = \left(\int_{\Omega(x)} |x|^p \, d\mu \right)^{1/p}, \qquad x \in \mathbf{S},\ \mu(\Omega(x)) > 0,$$

and for $p \geq 1$ the functional $\mathbf{p}_{\phi,\psi}$ is the $\mathbf{L}^p$-norm.

In the present paper we show that, under some general conditions, if $m: (0,\infty) \to (0,\infty)$ is a function such that

$$\mathbf{p}_{\phi,\psi}(tx) \leq m(t)\mathbf{p}_{\phi,\psi}(x), \qquad x \in \mathbf{S}_+,\ t > 0,$$

then m is multiplicative, and

$$\mathbf{p}_{\phi,\psi}(tx) = m(t)\mathbf{p}_{\phi,\psi}(x), \qquad x \in \mathbf{S},\ t > 0,$$

i.e., the functional $\mathbf{p}_{\phi,\psi}$ is *m-positively homogeneous*. Assuming that the underlying measure space (Ω, Σ, μ) has at least two sets of finite and positive measure, we show that the functional $\mathbf{p}_{\phi,\psi}$ is m-positively homogeneous if, and only if, the functions m, $\phi(\phi^{-1}(1)t)$, $\psi/\psi(1)$ are multiplicative, and $\psi(t) = \psi(1)m(\phi^{-1}(t)/\phi^{-1}(1))$, $t > 0$. This characterization simplifies if we assume that $\phi(1) = \psi(1) = 1$. If, moreover, arbitrary two functions chosen from the set $\{m, \phi, \psi\}$ satisfy some modest regularity assumptions then the functional $\mathbf{p}_{\phi,\psi}$ is m-positively homogeneous if, and only if, there exist real numbers p and q, $p \neq 0 \neq q$, such that $m(t) = t^p$, $\phi(t) = t^q$, and $\psi(t) = t^{p/q}$ for all $t > 0$. Taking $m(t) = t$ we obtain some earlier results of Zaanen [7], Wnuk [6], and the first named author of the present paper [3], as the special cases.

1. AN AUXILIARY REMARK AND A LEMMA

Remark 1. Suppose that $\mu(\Omega) > 0$ and take an arbitrary $x \in \mathbf{S}_+$ such that $\mu(\Omega(x)) > 0$. Then there are the pairwise disjoint sets $A_1, \dots, A_n \in \Sigma$, of finite and positive measure, and $x_1, \dots, x_n > 0$, such that

$$x = \sum_{i=1}^{n} x_i \chi_{A_i},$$

(here χ_A stands for the characteristic function of A). By the definition of $\mathbf{p}_{\phi,\psi}$ we get

$$p_{\phi,\psi}(x) = \psi\left(\sum_{i=1}^{n} \phi(x_i)\mu(A_i) \right).$$

In the sequel the following lemma plays an essential role [3, Theorem 1]:

LEMMA 1. *Let $\phi:(0,\infty)\to(0,\infty)$ be an arbitrary bijection. Then the function $\phi\circ(t\phi^{-1})$ is additive for every fixed $t>0$ if, and only if, the function*

$$(0,\infty)\ni t\to\phi(\phi^{-1}(1)t)$$

is multiplicative.

Proof. Suppose that for every fixed $t>0$ the function $\phi\circ(t\phi^{-1})$ is additive. Since $\phi\circ(t\phi^{-1})$ is positive, it must be linear. Thus there exists a function $M:(0,\infty)\to(0,\infty)$ such that

$$\phi\left[t\phi^{-1}(u)\right]=M(t)u,\qquad u>0;\,t>0,$$

and, of course $M(t)=\phi[\phi^{-1}(1)t]$, $t>0$. Replacing t by s, we have

$$\phi\left[s\phi^{-1}(u)\right]=M(s)u,\qquad u>0;\,s>0.$$

Composing separately the functions on the left, and on the right-hand sides of the above equations gives

$$\phi\left[st\phi^{-1}(u)\right]=M(s)M(t)u,\qquad u>0;\,s,t>0.$$

On the other hand we also have

$$\phi\left[st\phi(u)\right]=M(st)u,\qquad u>0;\,s,t>0,$$

and, consequently,

$$M(st)=M(s)M(t),\qquad s,t>0,$$

which means that $M:(0,\infty)\to(0,\infty)$ is multiplicative.

Since

$$M(t)=\phi\left[t\phi^{-1}(1)\right],\qquad t>0,$$

M is bijective, and consequently, $M^{-1}:(0,\infty)\to(0,\infty)$, the inverse of the function M,

$$M^{-1}(t)=\frac{\phi(t)^{-1}}{\phi^{-1}(1)},\qquad t>0,$$

is multiplicative.

Suppose that $M(u):=\phi[\phi^{-1}(1)u]$, $u>0$ is multiplicative. Then so is its inverse,

$$M^{-1}(u)=\frac{\phi^{-1}(u)}{\phi^{-1}(1)},\qquad u>0$$

and, consequently, for a fixed arbitrary $t > 0$, and for all $u, v > 0$, we have

$$\begin{aligned}\phi\left[t\phi^{-1}(u+v)\right] &= \phi\left[\phi^{-1}(1)t\frac{\phi^{-1}(u+v)}{\phi^{-1}(1)}\right] \\ &= M\left[tM^{-1}(u+v)\right] = M(t)(u+v) = M(t)u + M(t)v \\ &= M(t)M\left[M^{-1}(u)\right] + M(t)M\left[M^{-1}(v)\right] \\ &= M\left[tM^{-1}(u)\right] + M\left[tM^{-1}(v)\right] \\ &= \phi\left[t\phi^{-1}(u)\right] + \phi\left[t\phi^{-1}(v)\right],\end{aligned}$$

which completes the proof.

2. MAIN RESULTS

To give a complete characterization of m-positively homogeneous functionals $\mathbf{p}_{\phi,\psi}$ we prove the following

PROPOSITION. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure and let, $\phi, \psi: (0,\infty) \to (0,\infty)$ be bijective. Suppose that $m: (0,\infty) \to (0,\infty)$ is a function such that*

$$m(t)m(t^{-1}) \le 1, \qquad t > 0. \tag{1}$$

If

$$\mathbf{p}_{\phi,\psi}(tx) \le m(t)\mathbf{p}_{\phi,\psi}(x), \qquad t > 0,\, x \in \mathbf{S}_+, \tag{2}$$

then

$$\mathbf{p}_{\phi,\psi}(tx) = m(t)\mathbf{p}_{\phi,\psi}(x), \qquad t > 0,\, x \in \mathbf{S}, \tag{3}$$

the functions m, $\phi(\phi^{-1})t)$, and $\psi/\psi(1)$ are multiplicative, and

$$\psi(t) = \psi(1)m\left(\frac{\phi^{-1}(t)}{\phi^{-1}(1)}\right), \qquad t > 0, \tag{4}$$

Conversely, if the functions m, $\phi(\phi^{-1}(1)t)$ are multiplicative, and ψ is given by (4), then

$$p_{\phi,\psi}(tx) = m(t)p_{\phi,\psi}(x), \qquad t > 0,\, x \in \mathbf{S}.$$

Proof. Replacing x by tx, and t by t^{-1} in inequality (2) gives

$$\left[m(t^{-1})\right]^{-1}\mathbf{p}_{\phi,\psi}(x) \le \mathbf{p}_{\phi,\psi}(tx), \qquad t > 0,\, x \in \mathbf{S}_+.$$

Hence, applying (1) and (2), we obtain

$$m(t)\mathbf{p}_{\phi,\psi}(x) \le \left[m(t^{-1})\right]^{-1}\mathbf{p}_{\phi,\psi}(x) \le \mathbf{p}_{\phi,\psi}(tx) \le m(t)\mathbf{p}_{\phi,\psi}(x),$$

for all $t > 0$, $x \in \mathbf{S}_+$, which proves (3). From (3) we have

$$\mathbf{p}_{\phi,\psi}(stx) = m(st)\mathbf{p}_{\phi,\psi}(x), \qquad s,t > 0,\ x \in \mathbf{S}_+,$$

and

$$\mathbf{p}_{\phi,\psi}(stx) = m(s)\mathbf{p}_{\phi,\psi}(tx) = m(s)m(t)\mathbf{p}_{\phi,\psi}(x), \qquad s,t > 0,\ x \in \mathbf{S}_+,$$

and, consequently,

$$m(st)\mathbf{p}_{\phi,\psi}(x) = m(s)m(t)\mathbf{p}_{\phi,\psi}(x), \qquad s,t > 0,\ x \in \mathbf{S}_+.$$

By the assumption there are two sets $A, B \in \Sigma$ such that $a := \mu(A)$ $b := \mu(B)$ are positive, finite, and $A \cap B = \varnothing$. Taking here $x = \chi_A$ gives $m(st) = m(s)m(t)$ for all $s,t > 0$, which proves the multiplicativity of m. Taking $x := \chi_A$ in (2), in making use of Remark 1, we get

$$\psi(a\phi(t)) = m(t)\psi(a\phi(1)). \tag{5}$$

It follows that

$$\psi(t) = c_0 m(\phi^{-1}(a^{-1}t)), \qquad t > 0, \tag{6}$$

where $c_0 := \psi(a\phi(1))$ is positive. Taking $x := u\chi_A + v\chi_B$ in (1) with arbitrary $u, v > 0$, and applying Remark 1 gives

$$\psi(a\phi(tu) + b\phi(tv)) = m(t)\psi(a\phi(u) + b\phi(v)),$$

which, by (6), can be written in the form

$$\begin{aligned} &m\left(\phi^{-1}\left[a^{-1}(a\phi(tu) + b\phi(tv))\right]\right) \\ &\qquad = m(t)m\left(\phi^{-1}\left[a^{-1}(a\phi(u) + b\phi(v))\right]\right). \end{aligned}$$

Hence, making use of the multiplicativity of m, we get

$$m\left(\phi^{-1}\left(\phi(tu) + a^{-1}b\phi(tv)\right)\right) = m\left(t\phi^{-1}\left(\phi(u) + a^{-1}b\phi(v)\right)\right),$$
$$u, v, t > 0.$$

Since ϕ and ψ are one-to-one, relation (5) implies that so is m. It follows that (for short we put $\alpha := a^{-1}b$)

$$\phi^{-1}\left(\phi(tu) + \alpha\phi(tv)\right) = t\phi^{-1}\left(\phi(u) + \alpha\phi(v)\right), \qquad u, v, t > 0.$$

Taking the value ϕ of both sides, and replacing u and v by $\phi^{-1}(u)$ and $\phi^{-1}(v)$, respectively, gives

$$\phi(t\phi^{-1}(u)) + \alpha\phi(t\phi^{-1}(v)) = \phi(t\phi^{-1}(u + \alpha v)), \qquad u, v, t > 0.$$

Putting $f_t := \phi \circ (t\phi^{-1})$ for an arbitrary fixed $t > 0$, we hence get

$$f_t(u + \alpha v) = f_t(u) + \alpha f_t(v), \qquad u, v > 0. \tag{7}$$

Since f_t takes positive values in $(0, \infty)$, we have $f_t(u + \alpha v) > f_t(u)$, $u, v > 0$. It follows that for every $t > 0$, the function f_t is strictly increasing, and, as a bijection of $(0, \infty)$, f_t is an (increasing) homeomorphism of $(0, \infty)$. In particular,

$$\lim_{u \to 0+} f_t(u) = 0.$$

Hence, letting u tend to 0 in (7), we obtain

$$f_t(\alpha v) = \alpha f_t(v), \qquad v > 0.$$

This allows us to write (7) in the form $f_t(u + \alpha v) = f_t(u) + f_t(\alpha v)$, or equivalently,

$$f_t(u + v) = f_t(u) + f_t(v), \qquad u, v > 0,$$

for all $t > 0$. By the definition of f_t we get

$$\phi(t\phi^{-1}(u + v)) = \phi(t\phi^{-1}(u)) + \phi(t\phi^{-1}(v)), \qquad u, v, t > 0.$$

By Lemma 1 the function $(0, \infty) \ni t \to \phi(ct)$, $c := \phi^{-1}(1)$, is multiplicative. Hence, applying in turn, formula (6), the multiplicativity of the function $c^{-1}\phi^{-1}$ (which is the inverse of the function $(0, \infty) \ni t \to \phi(ct)$), and then the same property of the function m, we obtain

$$\begin{aligned}
\psi(t) &= c_0 m(\phi^{-1}(a^{-1}t)) = c_0 m\big(c\big[c^{-1}\phi^{-1}(a^{-1}t)\big]\big) \\
&= c_0 m\big(c\big[(c^{-1}\phi^{-1}(a^{-1}))(c^{-1}\phi^{-1}(t))\big]\big) \\
&= c_0 m\big((\phi^{-1}(a^{-1}))(c^{-1}\phi^{-1}(t))\big) \\
&= c_0 m(\phi^{-1}(a^{-1}))m(c^{-1}\phi^{-1}(t))
\end{aligned}$$

for all $t > 0$. Setting here $t = 1$, and taking into account the definition of c, and $m(1) = 1$, we hence get

$$\psi(1) = c_0 m(\phi^{-1}(a^{-1})),$$

and, consequently,

$$\phi(t) = \psi(1)m(c^{-1}\phi^{-1}(t)), \qquad t > 0,$$

which proves formula (4). This completes the proof of the first part of our proposition.

Now take $x \in \mathbf{S}$ with $\mu(\Omega(x)) > 0$, and $t > 0$. Applying in turn the definitions of $\mathbf{p}_{\phi,\psi}$, the multiplicativity of $\phi(ct)$, $t > 0$, formula (4) for ψ, the multiplicativity of the function $c^{-1}\phi^{-1}$ which is the inverse function of the function $\phi(ct)$, the multiplicativity of m, and, finally, formula (4), we have

$$\begin{aligned}
\mathbf{p}_{\phi,\psi}(tx) &= \psi\left(\int_{\Omega(X)} \phi \circ |tx|\, d\mu\right) = \psi\left(\int_{\Omega(X)} \phi \circ |ct(c^{-1}x)|\, d\mu\right) \\
&= \psi\left(\int_{\Omega(X)} \phi(ct)\phi \circ |c(c^{-1}x|\, d\mu\right) = \psi\left(\phi(ct)\int_{\Omega(X)} \phi \circ |x|\, d\mu\right) \\
&= \psi(1)m\left[c^{-1}\phi^{-1}\left(\phi(ct)\int_{\Omega(X)} \phi \circ |x|\, d\mu\right)\right] \\
&= \psi(1)m\left[c^{-1}\phi^{-1}(\phi(ct))c^{-1}\phi^{-1}\left(\int_{\Omega(X)} \phi \circ |x|\, d\mu\right)\right] \\
&= \psi(1)m\left[tc^{-1}\phi^{-1}\left(\int_{\Omega(X)} \phi \circ |x|\, d\psi\right)\right] \\
&= \psi(1)m(t)m\left[c^{-1}\phi^{-1}\left(\int_{\Omega(X)} \phi \circ |x|\, d\mu\right)\right] = m(t)\mathbf{p}_{\phi,\psi}(x),
\end{aligned}$$

which completes the proof.

Remark 2. To prove that (1) and (2) imply (3) and the multiplicativity of m, it is enough to assume that the underlying measure space (Ω, Σ, μ) has at least one set of finite and positive measure.

As an immediate consequence of the proposition we obtain:

THEOREM 1. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $\phi, \psi, m\colon (0,\infty) \to (0,\infty)$ are functions such that ϕ and ψ are one-to-one and onto. Then*

$$\mathbf{p}_{\phi,\psi}(tx) = m(t)\mathbf{p}_{\phi,\psi}(x), \qquad t > 0,\ x \in \mathbf{S}_+,$$

if, and only if, the functions m, $\phi^{-1}/\phi^{-1}(1)$, and $\psi/\psi(1)$ are multiplicative, and

$$\psi(t) = \psi(1)m\left(\frac{\phi^{-1}(t)}{\phi^{-1}(1)}\right), \qquad t > 0.$$

Remark 3. It is easy to see that if m, $\phi^{-1}/\phi^{-1}(1)$, are multiplicative, and ψ is given by (4), then

$$\mathbf{p}_{\phi,\psi}(tx) = m(t)\mathbf{p}_{\phi,\psi}(x), \qquad t > 0, x \in \mathbf{S}.$$

If $\phi(1) = \psi(1) = 1$ then the formulation of Theorem 1 becomes simpler:

THEOREM 2. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $\phi, \psi, m\colon (0,\infty) \to (0,\infty)$, ϕ and ψ are bijective, and $\phi(1) = \psi(1) = 1$. Then*

$$\mathbf{p}_{\phi,\psi}(tx) = m(t)\mathbf{p}_{\phi,\psi}(x), \qquad t > 0, x \in \mathbf{S}_{+},$$

if, and only if, m, ϕ, and ψ are multiplicative, and $\psi = m \circ \phi^{-1}$.

3. A CHARACTERIZATION OF m-POSITIVE HOMOGENEOUS FUNCTIONALS $\mathbf{p}_{\phi,\psi}$ FOR m, ϕ, AND ψ SATISFYING SOME REGULARITY CONDITIONS

Applying Theorem 1 with $m(t) = t^p$, and some well-known properties of multiplicative functions, Aczél [1, p. 41] or Kuczma [2, p. 310], we can easily deduce Corollaries 1 and 2 presented below.

COROLLARY 1. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $\phi, \psi\colon (0,\infty) \to (0,\infty)$ are bijective and*

$$\mathbf{p}_{\phi,\psi}(tx) = t^p\mathbf{p}_{\phi,\psi}(x), \qquad t > 0, x \in \mathbf{S}_{+},$$

where $p \neq 0$ is a fixed real number. If one of the following conditions is satisfied:

(i) *ϕ or ψ is bounded above in a neighborhood of a point;*

(ii) *$\log \circ \phi$ or $\log \circ \psi$ is bounded below in a neighborhood of a point;*

(iii) *ϕ or ψ is measurable,*

then there exists a real $q \neq 0$ such that

$$\phi(t) = \phi(1)t^q, \qquad \psi(t) = \psi(1)t^{p/q}, \qquad t > 0.$$

Remark 4. The above corollary remains true if we replace the functions ϕ and ψ in the conditions (i)–(iii), by ϕ^{-1} and ψ^{-1}.

In fact the following more general result holds true:

COROLLARY 2. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $m, \phi, \psi: (0, \infty) \to (0, \infty)$, ϕ and ψ are bijective, and*

$$\mathbf{p}_{\phi, \psi}(tx) = m(t)\mathbf{p}_{\phi, \psi}(x), \qquad t > 0, x \in \mathbf{S}_+.$$

If there is two-element subset $A \subset \{m, \phi, \psi\}$ such that each function from A is:

bounded above in a neighbourhood of a point,
or bounded below by a positive constant in a neighbourhood of a point,
or measurable,

then there exist real numbers p and q, $p \neq 0 \neq q$, such that

$$m(t) = t^p, \qquad \phi(t) = \phi(1)t^q \qquad \psi(t) = \psi(1)t^{p/q}, \qquad t > 0.$$

Remark 5. Corollary 2 remains true if we replace the functions ϕ and ψ in the conditions (i)–(iii) by ϕ^{-1} and ψ^{-1}.

Remark 6. The regularity assumptions in the above corollaries cannot be relaxed by imposing the following continuity type condition: for every $t_n \in \mathbf{R}$, $x_n \in \mathbf{S}$, if one of the sequences is bounded and the remaining tends to 0, then

$$\lim_{n \to \infty} \mathbf{p}_{\phi, \psi}(t_n x_n) = 0.$$

To see this it is enough to take a discontinuous multiplicative ϕ, and $\psi = \phi^{-1}$.

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