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KAHLIG, P.; MATKOWSKI, J.

Functional Equations Involving the Logarithmic Mean

Die Arbeit behandelt einige jüngere Ergebnisse über Funktionalgleichungen, die mit dem logarithmischen Mittel zusammenhängen, das bei einem Wärmeleitungsproblem auftritt. Indem die Funktionalgleichung auf alternative Weise neu interpretiert wird, kann eine nichttriviale Lösung gefunden werden.

The paper deals with some recent results concerning a functional equation involving the logarithmic mean which occurs in a heat conduction problem. By reinterpreting the functional equation in an alternative way, a nontrivial solution can be found.

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1. Introduction

The present paper was motivated by a consideration of the use of various means in meteorology, for instance:

(i) The *arithmetic* mean is traditionally used for averaging certain climate data. – A functional equation connecting two arithmetic means in the form

$$p\left(\frac{x+y}{2}\right) = \frac{p(x)+p(y)}{2}, \qquad x, y \in \mathbb{R} := (-\infty, \infty) \tag{1}$$

(Jensen's equation, cf. [1], [2]), may describe the hydrostatic pressure of an incompressible liquid as a function of depth, leading to the continuous solution (pressure-depth relation) $p(x) = p_0 + cx$, $x \in \mathbb{R}$ (p_0, c positive real constants).

(ii) A variant of Jensen's equation, connecting arithmetic and *geometric* mean, can describe the pressure-height relationship in a compressible isothermal gas atmosphere:

$$p\left(\frac{x+y}{2}\right) = [p(x)\,p(y)]^{1/2}, \qquad x, y \in \mathbb{R}, \tag{2}$$

yielding the continuous solution (barometric height formula)

$$p(x) = p_0 \exp(-x/H), \qquad x \in \mathbb{R} \tag{3}$$

(p_0, H positive real constants).

Note: Equivalent (symmetric) *Pexider equations* are $g(x+y) = p(x) + p(y)$ and $q(x+y) = p(x)\,p(y)$, respectively. Specializing $y = x$, the connection between g, q, p becomes evident: $g(2x) = 2p(x)$ and $q(2x) = [p(x)]^2$, hence $g(x) = 2p(x/2)$ and $q(x) = [p(x/2)]^2$, leading to (1) and (2). (For an alternative Pexider formulation cf. [5].)

(iii) The *logarithmic* mean is useful in some thermal transfer problems (cf. [11] sects. 8.7, 8.9), and appears also in the barometric height formula for a polytropic atmosphere (cf. [12] sect. 6.2):

$$p(x) = p_0 \exp(-cx/L), \qquad x \in \mathbb{R} \tag{4}$$

(p_0, c positive real constants), where L denotes the logarithmic mean temperature in the layer between heights x_0 and x (with corresponding absolute temperatures T_0 and T):

$$L = (T_0 - T)/\log(T_0/T), \qquad T_0, T \in (0, \infty) \tag{5}$$

(log = natural logarithm). [For isothermal conditions we get $T_0 = T = L$, thus reverting from (4) to (3) with a positive real constant $H = T_0/c$.]

In this vein, it appears expedient to investigate the applicability and usefulness of functional equations of the general form

$$f[M_1(x, y)] = M_2[f(x), f(y)] \tag{6}$$

(for certain domains, ranges, and regularity conditions), where M_1 and M_2 denote two given arbitrary means. If M_1, M_2 are *quasi-arithmetic* means, according to numerous known results (cf. [1], p. 281, [2], pp. 245–252), equation (6) has a nontrivial family of solutions. However, if one of the means is not quasi-arithmetic, only few results have been published.

In particular, M. Hosszú [4] proved that if M_1 is the logarithmic mean, and M_2 the arithmetic mean, the equation (6) in the form

$$f\left(\frac{x-y}{\log(x/y)}\right) = \frac{f(x)+f(y)}{2}, \qquad x, y > 0, \tag{7}$$

which occurs in a heat conduction problem (cf. [14] where the equation appears for the first time), has no nontrivial differentiable and strictly monotonic solutions (see also [13]). In section 2 we prove some new supplementary results showing a significant structural difference between arithmetic and logarithmic means; moreover, we strongly improve the original theorem of Hosszú. In section 3 we deal with the well-known Schröder functional equation $\varphi \circ h = \frac{1}{2}\varphi$, where $h(x) = (x-1)/\log x$, which in a natural way arises from equation (7), and we prove the basic Lemma 4. In section 4 we treat equation (7) as an infinite system of functional equations in a single variable with the parameter $\alpha := y$, interpretable as inhomogeneous Schröder equations. Applying Lemma 4, we give the general form of solutions which are differentiable at the point $x = \alpha$.

2. An improvement of a theorem of Hosszú

We begin with recalling the following

Definition 1: A function $L\colon (0,\infty)\times(0,\infty) \to (0,\infty)$,

$$L(x,y) = \frac{x-y}{\log x - \log y} \quad \text{for} \quad x \neq y; \qquad L(x,y) = x \quad \text{for} \quad x = y,$$

is said to be the *logarithmic mean* (cf. [3], p. 345).

Some properties: It is easy to see that

$$\min\{x,y\} \leqq L(x,y) \leqq \max\{x,y\}, \qquad x, y > 0,$$

and

$$\min\{x,y\} < L(x,y) < \max\{x,y\}, \qquad x, y > 0; \quad x \neq y.$$

Moreover, for every fixed x, the function of the second variable $L(x,.)$, namely $y \mapsto L(x,y)$, $y > 0$, is an increasing homeomorphism of $(0,\infty)$, i.e., $L(x,.)$ is strictly increasing, continuous, and

$$\lim_{y\to 0} L(x,y) = 0, \qquad \lim_{y\to\infty} L(x,y) = \infty.$$

The symmetry relation $L(x,y) = L(y,x)$, $x, y > 0$, implies that for every fixed $y > 0$ the function $L(.,y)$ is an increasing homeomorphism of $(0,\infty)$. It follows, in particular, that L has the *mean value property*, i.e., that for every interval $I \subset (0,\infty)$, we have $L(I\times I) = I$.

Remark 1: Hosszú [4] does not formulate his results in a precise way. In fact he claims that *the only differentiable solution of the functional equation* (7) *is the trivial constant* (cf. also [1], p. 82). But in his (indirect) argument the expression $\log f'(x) - \log f'(y)$ appears, i.e., he tacitly assumes that $f'(x)$ is positive for all x. As this expression can be written in the form $\log[f'(x)/f'(y)]$, his argument also works when $f'(x)$ is negative for all $x > 0$. Actually, assuming that f is defined on $(0,\infty)$, Hosszú proves the following

Theorem: *Suppose that* $f\colon (0,\infty) \to \mathbb{R}$ *is differentiable, and* $f'(x) > 0$ *for all* $x > 0$ *(or* $f'(x) < 0$ *for all* $x > 0$*). Then* f *is not a solution of* (7).

Of course this result is weaker than Hosszú's claim. In the following we shall prove the more general Theorem 1 which shows that there is a significant structural difference between quasi-arithmetic and logarithmic means. (In particular it shows that the second one is not quasi-arithmetic.) Moreover, after introducing Lemmas 1 to 3, we can prove Theorem 2 which is the main result of this section.

Theorem 1: *Let* $I \subseteq (0,\infty)$ *and* $J \subseteq \mathbb{R}$ *be arbitrary intervals. There is no bijective function* $g\colon I \to J$ *such that*

$$g\left(\frac{x-y}{\log x - \log y}\right) = \frac{g(x)+g(y)}{2}, \qquad x, y \in I. \tag{8}$$

Proof: Equation (8) can be written in the equivalent form

$$g^{-1}\left(\frac{g(x)+g(y)}{2}\right) = \frac{x-y}{\log x - \log y}, \qquad x, y \in I. \tag{9}$$

Suppose, for an indirect argument, that such a bijection exists. Take arbitrary $a, b, c, d \in I$, and put

$$x := g^{-1}\left(\frac{g(a) + g(b)}{2}\right), \qquad y := g^{-1}\left(\frac{g(c) + g(d)}{2}\right).$$

Since J is an interval, and g a bijection of I onto J, we have $x, y \in I$. Substituting x and y into (9) gives the identity

$$g^{-1}\left(\frac{g(a) + g(b) + g(c) + g(d)}{4}\right) = \left(\frac{a-b}{\log a - \log b} - \frac{c-d}{\log c - \log d}\right) \Big/ \left(\log\frac{a-b}{\log a - \log b} - \log\frac{c-d}{\log c - \log d}\right)$$

for all $a, b, c, d \in I$. Since

$$g^{-1}\left(\frac{g(a) + g(b) + g(c) + g(d)}{4}\right) = g^{-1}\left(\frac{g(a) + g(c) + g(b) + g(d)}{4}\right),$$

we hence get

$$\left(\frac{a-b}{\log a - \log b} - \frac{c-d}{\log c - \log d}\right) \Big/ \left(\log\frac{a-b}{\log a - \log b} - \log\frac{c-d}{\log c - \log d}\right)$$
$$= \left(\frac{a-c}{\log a - \log c} - \frac{b-d}{\log b - \log d}\right) \Big/ \left(\log\frac{a-c}{\log a - \log c} - \log\frac{b-d}{\log b - \log d}\right)$$

for all $a, b, c, d \in I$. By the analyticity of the function log in $(0, \infty)$ this equality would hold true for all a, b, c, d. This, however, is not the case which can be easily checked, taking for instance $a = e^{50}$, $b = e^{40}$, $c = e^{10}$, $d = e$: the difference between the left and the right hand sides is about 1.3×10^{15}. □

Lemma 1: *Let $I \subset (0, \infty)$ be an open interval. If $f : I \to \mathbb{R}$ is continuous at least at one point and satisfies the functional equation*

$$f\left(\frac{x-y}{\log x - \log y}\right) = \frac{f(x) + f(y)}{2}, \qquad x, y \in I, \tag{10}$$

then f is continuous at every point of I.

Proof: Suppose that f is continuous at a point $z \in I$, and let $x \in I$, $x \neq z$, be arbitrarily fixed. By the mean value property of L, there exists a $y \in I$ such that $x = L(z, y)$. Since the function $L(z, .)$ is an increasing homeomorphism of $(0, \infty)$, the point y is uniquely determined (see "Some properties" above). Take an arbitrary sequence $x_n \in I$ such that $\lim_{n\to\infty} x_n = x$. Again, by the above properties of L, for every positive integer n, there exists a uniquely determined $z_n \in I$ such that $x_n = L(z_n, y)$. Since the function $L(., y)$ is a homeomorphism of $(0, \infty)$, $L(z, y) = x$, and $\lim_{n\to\infty} x_n = x$, it follows that $\lim_{n\to\infty} z_n = z$. By the functional equation we have

$$f(x_n) = f(L(z_n, y)) = \frac{f(z_n) + f(y)}{2}, \qquad n \in \mathbb{N}.$$

Hence, by the continuity of f at the point z, letting $n \to \infty$, and making use of the functional equation, we have

$$\lim_{n\to\infty} f(x_n) = \frac{f(z) + f(y)}{2} = f(L(z, y)) = f(x),$$

which proves that f is continuous at the point x. □

Lemma 2: *Let $I \subset (0, \infty)$ be an open interval. Suppose that $f : I \to \mathbb{R}$ is a solution of the functional equation (10). If f is constant on a nonempty open subinterval of I, then f is constant on I.*

Proof: Let $I = (\alpha, \beta)$. Suppose that there is a $c \in \mathbb{R}$, and a nonempty open interval $(a, b) \subset I$ such that $f(x) = c$ for all $x \in (a, b)$. We can assume that (a, b) is maximal, i.e., if $J \subset I$ is an open interval such that f is constant on J and $(a, b) \subset J$, then $J = (a, b)$. Because every constant function satisfies equation (10), we may also assume, without any loss of generality, that $c = 0$.

For an indirect argument suppose that $I \setminus (a, b) \neq \emptyset$. Setting in equation (10) $x \in I$, $y \in (a, b)$, we get

$$f\left(\frac{x-y}{\log x - \log y}\right) = \frac{f(x)}{2}, \qquad x > 0, \quad y \in (a, b). \tag{11}$$

We shall need the following inequality (cf. [3], p. 348):

$$\sqrt{xy} < \frac{x-y}{\log x - \log y} < \frac{x+y}{2}, \qquad x, y > 0, \quad x \neq y. \tag{12}$$

Suppose first that $b < \beta$, and put $\beta_1 := \min\{b + (b-a)/2, \beta\}$. For $y = (a+b)/2, b \leqq x < \beta_1$, by (12) we have $a < (x-y)/(\log x - \log y) < b$. Hence, setting $y = (a+b)/2$, and arbitrary $x \in [b, \beta_1)$ in (11) gives

$$f(x) = 2f\left(\frac{x-y}{\log x - \log y}\right) = 0\,, \qquad x \in \left[b, b + \frac{b-a}{2}\right),$$

which contradicts the maximality of the interval (a, b). Thus $b = \beta$. In a similar way we can show that $a = \alpha$. □

Lemma 3: *Let $I \subset (0, \infty)$ be an open interval. Suppose that $f: I \to \mathbb{R}$ is a continuous solution of the functional equation* (10). *If there are $a, b \in I$, $a \neq b$, such that $f(a) = f(b)$, then f is constant on I.*

Proof: We may assume that $a < b$. Put $C := \{x \in [a, b] : f(x) = f(a)\}$. We shall show that $C = [a, b]$. For an indirect proof suppose that $[a, b] \setminus C$ is nonempty. By the continuity of f the set C is closed. It follows that there exists a nonempty open maximal interval $(c, d) \subset [a, b] \setminus C$. Thus $f(c) = f(d) = f(a)$, i.e. $c, d \in C$. From (10) we have $f\left(\frac{c-d}{\log c - \log d}\right) = \frac{f(c) + f(d)}{2} = f(a)$. This is a contradiction because $c < \frac{c-d}{\log c - \log d} < d$, and consequently, $\frac{c-d}{\log c - \log d} \notin C$. Now, by Lemma 2, $f(x) = f(a)$ for all $x \in I$, which was to be shown. □

Theorem 2: *Let $I \subset (0, \infty)$ be a nonempty open interval, and suppose that $f: I \to \mathbb{R}$ satisfies the functional equation*

$$f\left(\frac{x-y}{\log x - \log y}\right) = \frac{f(x) + f(y)}{2}\,, \qquad x, y \in I\,.$$

If f is continuous at least at one point, then f is a constant function.

Proof: By Lemma 1 the function f is continuous on I. In view of Theorem 1 the function f cannot be one-to-one because, by its continuity, it would be a bijection of I onto the interval $J = f(I)$. Thus there are $a, b \in I$ such that $f(a) = f(b)$. By Lemma 3, f is constant on I. □

3. A special Schröder functional equation

In this section, applying some known facts from the theory of iterative functional equations (cf. for instance [6], [7], [8]), we consider a special Schröder functional equation $\varphi \circ h = \frac{1}{2}\varphi$, where the given function h is described below.

Remark 2: Define $h: (0, \infty) \to \mathbb{R}$ by the formula

$$h(x) := \frac{x-1}{\log x} \quad \text{for} \quad x > 0\,, \quad x \neq 1\,; \qquad h(x) = 1 \quad \text{for} \quad x = 1\,, \tag{13}$$

and note some of its properties:

(A) $h: (0, \infty) \to (0, \infty)$ is an increasing and concave homeomorphism, mapping $(0, \infty)$ onto itself. Moreover, $h(1) = 1$,

$$h(0+) := \lim_{x \to 0+} h(x) = 0\,, \qquad h(\infty) := \lim_{x \to +\infty} h(x) = +\infty\,,$$

$$2(\sqrt{x} - 1) < 2[h(x) - 1] < x - 1\,, \qquad x > 0\,.$$

(B) h is analytic in $(0, \infty)$, and $h'(1) = \frac{1}{2}$,

$$h'(0+) := \lim_{x \to 0+} h'(x) = \infty\,, \qquad h'(\infty) := \lim_{x \to \infty} h'(x) = 0\,.$$

(C) For every $x > 0, x \neq 1, 0 < [h(x) - 1]/(x - 1) < 1$, or, equivalently, $x < h(x) < 1, x \in (0, 1)$, and $1 < h(x) < x$, $x \in (1, \infty)$.

(D) Let h^n stand for the n-th iterate of h. Then for every $x > 0$, $\lim_{n \to \infty} h^n(x) = 1$, and the convergence is uniform on every compact subset of $(0, \infty)$. Moreover, $x^{2^{-n}} \leqq h^n(x) \leqq 2^{-n}(x + 2^n - 1)$, $x > 0$, $n \in \mathbb{N}$ (equality holding only for $x = 1$).

Proof: According to (12), arithmetic and geometric means constitute bounds for the logarithmic mean. Fixing $y = 1$ gives the inequality of (A). The remaining statements are easy to verify. □

Remark 3: Let us note that the function h is *convex conjugate* (cf. [10]); it means that $h = h^*$ where $h^*(x) := xh(x^{-1})$, $x > 0$.

Now we can formulate the following

Lemma 4: *Let* $h:(0,\infty)\to\mathbb{R}$ *be defined by* (13).

(A) *If* $\varphi:(0,\infty)\to\mathbb{R}$ *is a solution of the functional equation*

$$\varphi[h(x)] = \tfrac{1}{2}\,\varphi(x)\,,\qquad x>0\,, \tag{14}$$

such that φ *is differentiable at* 1, *then the sequence of functions* $V_n:(0,\infty)\to\mathbb{R}$ *defined by*

$$V_n(x) := 2^n[h^n(x)-1]\,,\qquad x>0\,,\quad n\in\mathbb{N}\,, \tag{15}$$

converges for every $x>0$, *and*

$$\varphi(x) = cV(x)\,,\qquad x>0\,, \tag{16}$$

where $c:=\varphi'(1)$, *and* $V:(0,\infty)\to\mathbb{R}$ *is defined by the formula*

$$V(x) := \lim_{n\to\infty} V_n(x)\,,\qquad x>0\,. \tag{17}$$

The function V *is analytic, increasing, and concave in* $(0,\infty)$, *and*

$$\log x \leqq V(x) \leqq V_n(x) \leqq x-1\,,\qquad x>0\,,\quad n\in\mathbb{N}\,. \tag{18}$$

(B) *The formula* (16) *gives a unique one-parameter family of solutions of equation* (14) *which are differentiable at* 1. *In fact, these solutions are analytic in* $(0,\infty)$. *Moreover,* φ *is increasing and concave for* $c>0$, *and decreasing and convex for* $c<0$.

Proof: (A) From Remark 2(D) we get $2^n(x^{2^{-n}}-1)\leqq V_n(x)\leqq x-1$, $x>0$, $n\in\mathbb{N}$. Since (cf. [6], p. 161) $\lim_{n\to\infty} 2^n(x^{2^{-n}}-1) = \log x$, $x>0$, the sequence $V_n(x)$ is bounded. The inequality $2[h(x)-1]\leqq x-1$, $x>0$, in Remark 2(A) implies that $V_{n+1}(x)\leqq V_n(x)$, $x>0$, $n\in\mathbb{N}$, i.e., the sequence of functions V_n is decreasing. It follows that $V(x)$ exists and relations (17) and (18) hold true.

Suppose that $\varphi:(0,\infty)\to\mathbb{R}$ is differentiable at 1 and satisfies (14). Then $\varphi(1)=0$, and the function $\psi:(0,\infty)\to\mathbb{R}$,

$$\psi(x) := (x-1)^{-1}\,\varphi(x)\,,\qquad x\neq 1\,;\qquad \psi(1) := \varphi'(1)\,,$$

is a solution, continuous at 1, of the functional equation

$$\psi(x) = 2\,\frac{h(x)-1}{x-1}\,\psi[h(x)]\,,\qquad x>0\,,\quad x\neq 1\,.$$

Hence, by induction, we obtain

$$\psi(x) = 2^n\,\frac{h^n(x)-1}{x-1}\,\psi[h^n(x)]\,,\qquad x>0\,,\quad x\neq 1\,,\quad n\in\mathbb{N}\,,$$

which can be written in the equivalent form

$$\varphi(x) = \psi[h^n(x)]\,V_n(x)\,,\qquad x>0\,,\quad n\in\mathbb{N}\,. \tag{19}$$

By the continuity of ψ at 1, and the first part of Remark 2(D), we have $\lim_{n\to\infty}\psi[h^n(x)] = \psi(1) = \varphi'(1)$, $x>0$. Consequently, letting $n\to\infty$ in (19), we get $\varphi(x) = \psi(1)\lim_{n\to\infty} V_n(x) = cV(x)$, $x>0$.

(B) In view of Remark 2(B) and 2(C), the function h satisfies all the assumptions of Koenigs' theorem (cf. [6], p. 140). Therefore, the solution φ given by (16) is analytic in a neighbourhood of the point 1, the fixed point of the function h. Now (14), the global analyticity of h, and the first of the inequalities of Remark 2(C) easily imply that φ is analytic in $(0,\infty)$. – The remaining statements are consequences of well-known results (cf. [6], p. 142, Theorem 6.7, and p. 143, Theorem 6.8). □

Remark 4: Note that $V(1)=0$ and $V'(1)=1$. Therefore, (16) allows to interpret the real constant c as $\varphi'(1)$.

Remark 5: Let us note that (16) gives a unique one-parameter family of concave solutions for $c\geqq 0$, and convex for $c\leqq 0$, separately in the intervals (0, 1) and $(1,\infty)$ (cf. [6], pp. 142, 143).

Remark 6: Note that (cf. [6] p. 143) the solutions φ of (14) can also be presented in the following way:

$$\varphi(x) = c\lim_{n\to\infty}\frac{h^n(x)-1}{h^n(x_0)-1}\,,\qquad x>0\,,$$

where $x_0>0$, $x_0\neq 1$, is arbitrarily fixed. This form of the solution is termed the *principal* one.

4. A reinterpretation of functional equation (7)

As a consequence of the negative results of Theorems 1 and 2, we will formulate the problem in an alternative (but necessarily weaker) way: we will look at (7) as a functional equation in a single variable with a parameter, interpretable as an inhomogeneous Schröder equation. In fact, (7) is an infinite system of inhomogeneous Schröder equations. – The main result reads as follows.

Theorem 3: *Let $\alpha > 0$ be fixed. If $\varphi_\alpha : (0, \infty) \to \mathbb{R}$ is a solution of the functional equation*

$$\varphi_\alpha\left(\frac{x-\alpha}{\log(x/\alpha)}\right) = \frac{1}{2}\left[\varphi_\alpha(x) + \varphi_\alpha(\alpha)\right], \qquad x > 0, \tag{20}$$

such that φ_α is differentiable at the point α, then there exist real constants c and k such that

$$\varphi_\alpha(x) = cV(x/\alpha) + k, \qquad x > 0, \tag{21}$$

where $V:(0, \infty) \to \mathbb{R}$ is defined by (17) and (15). Equation (21) gives a unique two-parameter family of solutions of (20) which are analytic in $(0, \infty)$; moreover, φ_α is monotonic and either concave or convex.

Proof: Suppose that φ_α satisfies (20). Writing this equation in the form

$$\varphi_\alpha\left(\alpha\,\frac{(x/\alpha)-1}{\log(x/\alpha)}\right) = \frac{1}{2}\left[\varphi_\alpha(\alpha(x/\alpha)) - \varphi_\alpha(\alpha)\right], \qquad x > 0,$$

we see that the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(x) := \varphi_\alpha(\alpha x) - \varphi_\alpha(\alpha)$, $x > 0$, satisfies the functional equation

$$\varphi\left(\frac{(x/\alpha)-1}{\log(x/\alpha)}\right) = \frac{1}{2}\,\varphi(x/\alpha), \qquad x > 0.$$

Replacing x/α by x, and setting $h(x) := (x-1)/\log x$, $x > 0$, with $h(1) := 1$, we get $\varphi[h(x)] = \frac{1}{2}\varphi(x)$, $x > 0$, which means that φ is a solution of (14). – Suppose that φ_α is differentiable at the point α. Then φ is differentiable at 1, and by Lemma 4, there exists a constant c such that $\varphi(x) = cV(x)$, $x > 0$, where V is defined by (17) and (15). Hence $\varphi_\alpha(\alpha x) = cV(x) + \varphi_\alpha(\alpha)$, $x > 0$, and, consequently, $\varphi_\alpha(x) = cV(x/\alpha) + \varphi_\alpha(\alpha)$, $x > 0$. Since the function $x \mapsto cV(x/\alpha)$, $x > 0$, vanishes at the point $x = \alpha$, and every constant function satisfies equation (20), for $\varphi_\alpha(\alpha)$ one can take an arbitrary constant $k \in \mathbb{R}$. – The remaining statements of the theorem follow from Lemma 4. □

Remark 7: Relation (21), for every fixed $\alpha > 0$, represents a two-parameter family of solutions. Replacing k in this formula by the constant $k + cV(\alpha)$ we can write (21) in the form $\varphi_\alpha(x) = c[V(x/\alpha) + V(\alpha)] + k$, $x > 0$. This representation is especially interesting: defining a two-place function $F : (0, \infty) \times (0, \infty) \to \mathbb{R}$ by

$$F(x, \alpha) := V(x/\alpha) + V(\alpha), \qquad x, \alpha > 0,$$

it turns out that, under some general conditions, the one-parameter family of graphs of the functions $x \mapsto F(x, \alpha)$, $x > 0$ (where α is the parameter) has an envelope which is the graph of the function $E : (0, \infty) \to \mathbb{R}$,

$$E(x) = 2V(x^{1/2}), \qquad x > 0.$$

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Addresses: Prof. Dr. Peter Kahlig, Inst. of Meteorology & Geophysics, University, A-1190 Vienna, Austria; Prof. Dr. Janusz Matkowski, Dept. of Mathematics, Technical University, PL-43-309 Bielsko-Biała, Poland