

NONLINEAR CONTRACTIONS ON SEMIMETRIC SPACES

J. JACHYMSKI, J. MATKOWSKI, and T. ŚWIĄTKOWSKI*

Abstract. Let (X, d) be a Hausdorff semimetric (d need not satisfy the triangle inequality) and d -Cauchy complete space. Let f be a selfmap on X , for which $d(fx, fy) \leq \phi(d(x, y))$, $(x, y \in X)$, where ϕ is a non-decreasing function from \mathbf{R}_+ , the nonnegative reals, into \mathbf{R}_+ such that $\phi^n(t) \rightarrow 0$, for all $t \in \mathbf{R}_+$. We prove that f has a unique fixed point if there exists an $r > 0$, for which the diameters of all balls in X with radius r are equibounded. Such a class of semimetric spaces includes the Frechet spaces with a regular ecart, for which the Contraction Principle was established earlier by M. Cicchese [5], however, with some further restrictions on a space and a map involved. We also demonstrate that for maps f satisfying the condition $d(fx, fy) \leq \phi(\max\{d(x, fx), d(y, fy)\})$, $(x, y \in X)$ (the Bianchini [2] type condition), a fixed point theorem holds under substantially weaker assumptions on a distance function d .

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*The author passed away in October 30, 1994

1. Introduction. A *distance function* for a set X is a function d from $X \times X$ into \mathbf{R}_+ , the nonnegative reals, such that $d(x, y) = 0$ iff $x = y$, and $d(x, y) = d(y, x)$ for all $x, y \in X$. A distance function is also called a *symmetric*. The space (X, d) in which limiting points are defined in the usual way is called an *E-space*. The idea of *E-spaces* goes back to Frechet and Menger. The pioneer works in this setting were the papers of E. W. Chittenden [4] and W. A. Wilson [17].

In every symmetric space (X, d) one may introduce a topology τ_d by defining the family of closed sets as follows: a set $A \subseteq X$ is closed iff for any $x \in X$, $d(x, A) = 0$ implies $x \in A$, where

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

A topological space (X, τ) is *symmetrizable* iff there exists a symmetric d for which τ_d coincides with τ . A space (X, τ) is *semimetrizable* iff there is a distance function d such that for any $A \subseteq X$, $\bar{A} = \{x \in X : d(x, A) = 0\}$. In this case d is said to be a semimetric. In other words, without involving a topology, d is a semimetric if the operator

$$cl(A) := \{x \in X : d(x, A) = 0\}, \text{ for } A \subseteq X,$$

is the closure operator (it suffices here that cl is idempotent, i.e., $cl(cl(A)) = cl(A)$ for all $A \subseteq X$). For a discussion of the differences between a semimetric space and a symmetric space, see [1] and the references in [3].

Further, a symmetric or semimetric space (X, d) is *d-Cauchy complete* if every *d*-Cauchy sequence is τ -convergent (a sequence $\{x_n\}_{n=1}^\infty$ is *d*-Cauchy if given $\epsilon > 0$, there is a $k \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq k$). We emphasize here that there are several concepts of completeness in semimetric spaces (see [15], [8]), but for our purposes we shall employ only the above concept. Similarly, an *E-space* (X, d) is *complete* if every *d*-Cauchy sequence $\{x_n\}_{n=1}^\infty$ is *d-convergent*, i.e., $d(x_n, x_0) \rightarrow 0$, for some $x_0 \in X$. Since in semimetrizable spaces *d*-convergence coincides with τ -convergence (see, e.g., [8]), we may conclude that a semimetric space (X, d) is *d*-Cauchy complete iff the *E-space* (X, d) is complete.

Our main purpose is to extend some fundamental metric fixed point theorems to a non-metric setting. Namely, we generalize Theorem 1.2 [13] of the second named author (see also [6], Theorem 3.2, or [14], Theorem 2) by considering selfmaps on some *d*-Cauchy complete semimetric spaces. This class of spaces is large enough to include the spaces (X, d) studied in [4] (called by Frechet spaces with a *regular ecart*), for which d is assumed to satisfy the following condition, a relaxation of the triangle inequality.

$$(1) \quad d(x, y) \leq \epsilon(\max\{d(x, z), d(z, y)\}), \quad \text{for } x, y, z \in X,$$

where a function $\epsilon : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is such that $\lim_{t \rightarrow 0^+} \epsilon(t) = 0$. (Recently, a comprehensive study of such spaces with ϵ being linear has been made by the third named author [16] in connexion with studying the so-called small system convergence [12].). Our Theorem 1 generalizes an earlier result of M. Cicchese [5], who has considered the Banach contractions on a semimetric space with d satisfying a strengthened form of (1). Moreover, a restriction on a contractive constant was made in [5].

We also give an example of a fixed point free Banach contraction on a d -Cauchy complete semimetric space in order to demonstrate that an additional condition imposed on d in Theorem 1 cannot be omitted (see Example 2). On the other hand, our Theorem 2 shows that this condition is unnecessary if one considers a map f satisfying the inequality introduced by R. M. Bianchini [2].

$$(2) \quad d(fx, fy) \leq h \max\{d(x, fx), d(y, fy)\},$$

for an $h \in (0, 1)$ and all $x, y \in X$.

Then, however, a continuity argument must be used to ensure the existence of a fixed point (see Example 3 and Remark 3).

Finally, we would like to call the reader's attention to the recent papers [9], [10] and [11] of T. Hicks and B. E. Rhoades, in which the authors have obtained several fixed point theorems for maps on so-called d -complete topological spaces. Here a distance function d need not be even symmetric. However, they use a different concept of completeness: a space (X, d) is said to be (Σ) d -complete iff for any sequence $\{x_n\}_{n=1}^\infty$, $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}_{n=1}^\infty$ is τ -convergent. This notion led to obtain almost immediately an extension of the Contraction Principle and many other theorems to such spaces. However, in a semimetric setting, this concept of completeness is rather strong (see Proposition 2).

2. Preliminary results. We begin with the following simple extension of the Contraction Principle. The letter f^n denotes the n th iterate of a map f .

PROPOSITION 1. *Let (X, d) be a Hausdorff semimetric and d -Cauchy complete space and let f be a selfmap on X satisfying the Banach contractive condition:*

$$(3) \quad d(fx, fy) \leq hd(x, y), \quad \text{for an } h \in (0, 1) \text{ and } x, y \in X.$$

If (X, d) is bounded, i.e., $M := \sup\{d(x, y) : x, y \in X\} < \infty$, then f has a unique fixed point p , and for any $x \in X$, $\{f^n x\}_{n=1}^\infty$ converges to p .

Proof. Fix an $x \in X$. That $\{f^n x\}_{n=1}^\infty$ is d -Cauchy follows easily from the inequality

$$d(f^n x, f^{n+m} x) \leq h^n d(x, f^m x) \leq h^n M, \text{ for all } n, m \in \mathbf{N}$$

because of the convergence $h^n M \rightarrow 0$. By the completeness, there is a $p \in X$ such that $\{f^n x\}_{n=1}^\infty$ τ -converges to p . Since d is a semimetric, (3) implies that f is τ -continuous. Therefore, $\{f^{n+1} x\}_{n=1}^\infty$ τ -converges to fp . Since (X, d) is Hausdorff, we may infer that $p = fp$. Clearly, (3) guarantees the uniqueness of a fixed point. \square

The following example shows that under assumptions of Proposition 1 a space (X, d) need not be (Σ) d -complete.

EXAMPLE 1. Let $X := \mathbf{N}$. Define the function d by putting

$$d(n, n+1) := \frac{1}{2^n} =: d(n+1, n), \quad d(n, n) := 0, \quad \text{for } n \in \mathbf{N},$$

and $d(n, m) := 1$, for all $n, m \in \mathbf{N}$ with $|n - m| > 1$. Then d is the semimetric. Clearly, $\sum_{n=1}^\infty d(n, n+1) < \infty$ and the sequence $\{n\}_{n=1}^\infty$ is not convergent. Thus, (X, d) is not (Σ) d -complete. On the other hand, every d -Cauchy sequence is constant for sufficiently large n , hence convergent, so (X, d) is d -Cauchy complete.

Moreover, for a large class of semimetric spaces, (Σ) d -completeness implies d -Cauchy completeness.

PROPOSITION 2. Let (X, d) be a semimetric space satisfying Wilson's Axiom IV [17], i.e., given $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ and an x in X ,

$$d(x_n, x) \rightarrow 0 \text{ and } d(x_n, y_n) \rightarrow 0 \text{ imply that } d(y_n, x) \rightarrow 0.$$

If (X, d) is (Σ) d -complete, then (X, d) is d -Cauchy complete.

Proof. Let $\{x_n\}_{n=1}^\infty$ be a d -Cauchy sequence. Then there exists a subsequence $\{x_{k_n}\}_{n=1}^\infty$ such that $d(x_{k_n}, x_{k_{n+1}}) < \frac{1}{2^n}$. By hypothesis, $\{x_{k_n}\}_{n=1}^\infty$ is τ -convergent to an $x \in X$. Since d is a semimetric, we get $d(x_{k_n}, x) \rightarrow 0$. Simultaneously, $d(x_n, x_{k_n}) \rightarrow 0$ because of the Cauchy condition. So, by Axiom IV, we may infer that $d(x_n, x) \rightarrow 0$, which implies that $\{x_n\}_{n=1}^\infty$ is τ -convergent to x (this implication also holds if d is a symmetric). \square

The following example shows that Proposition 1 cannot be extended to unbounded semimetric spaces.

EXAMPLE 2. Let $X := \mathbb{N}$, $fn := n + 1$ for $n \in \mathbb{N}$, and

$$d(n, m) := \frac{|n - m|}{2^{\min\{n, m\}}}, \quad \text{for } n, m \in \mathbb{N}.$$

Then d is the semimetric. Let $\{x_n\}_{n=1}^\infty$ be a d -Cauchy sequence. Then $\{x_n\}_{n=1}^\infty$ is bounded; for otherwise, there is a subsequence $\{x_{k_n}\}_{n=1}^\infty$, $x_{k_n} \rightarrow \infty$, and then, for any $n \in \mathbb{N}$,

$$\lim_{m \rightarrow \infty} d(x_{k_n}, x_{k_m}) = \lim_{m \rightarrow \infty} \frac{|x_{k_n} - x_{k_m}|}{2^{x_{k_n}}} = \infty,$$

violating the Cauchy condition. Therefore, we may infer that $\{x_n\}_{n=1}^\infty$ is constant for sufficiently large n , since it is d -Cauchy. Thus (X, d) is d -Cauchy complete, but f has no a fixed point though it satisfies (3) with $h = \frac{1}{2}$.

Now, we give some equivalent formulations of a condition imposed on a function d in our Theorem 1 (see Section 3).

PROPOSITION 3. Let (X, d) be an E -space. The following conditions are equivalent.

(i): There exists an $r > 0$ such that

$$R := \sup\{\text{diam } K(x, r) : x \in X\} < \infty,$$

i.e., the diameters of open balls with radius r are equibounded.

(ii): There exist $\delta, \eta > 0$ such that, given $x, y, z \in X$,

$$d(x, z) + d(z, y) \leq \delta \text{ implies that } d(x, y) \leq \eta.$$

(iii): There do not exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ such that

$$d(x_n, z_n) \rightarrow 0, \quad d(z_n, y_n) \rightarrow 0 \text{ and } d(x_n, y_n) \rightarrow \infty.$$

Proof. To prove (i) implies (ii) it suffices to put $\delta := \frac{r}{2}$ and $\eta := R$. To prove (ii) implies (iii) suppose, on the contrary, there exist sequences as in (iii). Then $d(x_n, z_n) + d(z_n, y_n) \leq \delta$ for n large enough so, by (ii), $d(x_n, y_n) \leq \eta$, which contradicts the convergence $d(x_n, y_n) \rightarrow \infty$. Further, it is easy to verify the implication $\neg(i) \Rightarrow \neg(iii)$. \square

3. Nonlinear contractions on a semimetric space. Obviously, condition (iii) of Proposition 3 is satisfied if there do not exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ such that

$$d(x_n, z_n) \rightarrow 0, \quad d(z_n, y_n) \rightarrow 0 \quad \text{and} \quad d(x_n, y_n) \not\rightarrow 0.$$

By Theorem 1 of Wilson [17], the last condition holds iff (X, d) has a regular ecart. So, in particular, the following fixed point theorem may be applied to selfmaps on such a space.

THEOREM 1. *Let (X, d) be a Hausdorff semimetric and d -Cauchy complete space satisfying one (hence all) of conditions (i)–(iii) of Proposition 3. Let f be a selfmap on X , for which*

$$(4) \quad d(fx, fy) \leq \phi(d(x, y)), \quad \text{for all } x, y \in X,$$

where $\phi: \mathbf{R}_+ \mapsto \mathbf{R}_+$ is a non-decreasing function such that $\lim_{n \rightarrow \infty} \phi^n(t) = 0$, ($t \in \mathbf{R}_+$). Then f has a unique fixed point p and $f^n x \rightarrow p$, for all $x \in X$.

Proof. Assume that condition (ii) of Proposition 3 holds. Fix an $x \in X$. By (4) and the monotonicity of ϕ , we get that

$$(5) \quad d(f^n x, f^{n+m} x) \leq \phi^n(d(x, f^m x)), \quad \text{for all } n, m \in \mathbf{N}.$$

In particular, $d(f^n x, f^{n+1} x) \leq \phi^n(d(x, fx))$, which implies, by hypothesis, that $d(f^n x, f^{n+1} x) \rightarrow 0$. Therefore, there is a $k \in \mathbf{N}$ such that $d(f^k x, f^{k+1} x) \leq \min\{\frac{\delta}{2}, \eta\}$. Assume that $\phi(\eta) \leq \frac{\delta}{2}$. We shall apply induction with respect to n to show that, for all $n \in \mathbf{N}$,

$$(6) \quad d(f^k x, f^{k+n} x) \leq \eta.$$

By the definition of k , (6) holds for $n = 1$. Assume that (6) is satisfied for some $n \in \mathbf{N}$. Since $d(f^k x, f^{k+1} x) \leq \frac{\delta}{2}$ and

$$d(f^{k+1} x, f^{k+n+1} x) \leq \phi(d(f^k x, f^{k+n} x)) \leq \phi(\eta) \leq \frac{\delta}{2},$$

we get that $d(f^k x, f^{k+1} x) + d(f^{k+1} x, f^{k+n+1} x) \leq \delta$, which, by (ii), implies that $d(f^k x, f^{k+n+1} x) \leq \eta$, completing the induction. Hence and by (5), we may infer that

$$d(f^{k+n} x, f^{k+n+m} x) \leq \phi^n(\eta), \quad \text{for all } n, m \in \mathbf{N},$$

which easily yields the Cauchy condition for $\{f^n x\}_{n=1}^\infty$. Further, use the same argument as in the proof of Proposition 1 to obtain that $f^n x \rightarrow p = fp$. Thus the proof is completed if $\phi(\eta) \leq \frac{\delta}{2}$. If not, then, however, there exists a $j \in \mathbf{N}$ such that $\phi^j(\eta) \leq \frac{\delta}{2}$. Since the iterate f^j satisfies (4) with ϕ replaced

by ϕ^j , we may conclude by the preceding part of the proof, that f^j has a unique fixed point p and for all $x \in X$, $f^{jn}x \rightarrow p$ as $n \rightarrow \infty$. It is well-known that this implies that p is a unique fixed point of f and $f^n x \rightarrow p$, for all $x \in X$ (clearly, the proof of this fact in a metric setting remains valid for semimetrics). \square

REMARK 1. Theorem 1 generalizes Theorem 2 of Cicchese [5], who has imposed on d the condition

$$d(x, y) \leq \epsilon(d(x, z)) + kd(y, z), \quad \text{for all } x, y, z \in \mathbf{R}_+,$$

where $k \geq 1$, $\epsilon : [0, a) \mapsto \mathbf{R}_+$, ($a > 0$) and $\lim_{t \rightarrow 0^+} \epsilon(t) = 0$. By Theorem 1 [17], this condition is stronger than (1). Furthermore, Cicchese has assumed that f satisfies (3) with $h < \frac{1}{k}$.

REMARK 2. Theorem 1 can be carried over to a complete E -space (X, d) satisfying (i) of Proposition 3 and Wilson's Axiom III [17] given in our Theorem 2.

4. Bianchini's maps on an E -space. The following example shows that Proposition 1 cannot be extended to maps satisfying condition (2).

EXAMPLE 3. Let $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbf{N}\}$ and $fx := \frac{x}{4}$ for $x \neq 0$, $f0 := 1$. Further, let

$$d(0, 1) := 1 =: d(1, 0), \quad d(1, \frac{1}{n}) := \frac{2}{3} =: d(\frac{1}{n}, 1) \quad \text{for } n \geq 2, \quad (7)$$

$$d(1, 1) := 0 \quad \text{and} \quad d(x, y) := |x - y|, \quad \text{for } x, y \in X - \{1\}.$$

Then d is the semimetric. Let $\{x_n\}_{n=1}^\infty$ be a d -Cauchy sequence. Since $(X - \{1\}, d)$ is the complete metric space, it suffices to consider the case, in which there is a subsequence $\{x_{k_n}\}_{n=1}^\infty$ such that $x_{k_n} = 1$, for all $n \in \mathbf{N}$. Then $x_n = 1$ for sufficiently large n ; for otherwise, there is a subsequence $\{x_{m_n}\}_{n=1}^\infty$ such that $x_{m_n} \neq 1$ for all $n \in \mathbf{N}$ so, by (7), $d(x_{k_n}, x_{m_n}) \geq \frac{2}{3}$, ($n \in \mathbf{N}$), violating the Cauchy condition. Thus (X, d) is d -Cauchy complete.

Now, we verify condition (2). Let $X_0 := X - \{0, 1\}$. Then (X_0, d) is the metric space and $f|_{X_0}$ is the Banach contraction with the constant $h = \frac{1}{4}$. Hence, by the triangle inequality, $f|_{X_0}$ satisfies (2) with the constant $\frac{2h}{1-h}$ ($= \frac{2}{3}$). Further, for all $x \in X$, $d(f0, fx) \leq \frac{2}{3} = \frac{2}{3}d(0, f0)$, and $d(f1, fx) = \frac{1}{4} - \frac{1}{4}x < \frac{1}{9} = \frac{2}{3}d(1, f1)$, for $x \neq 0$. So f satisfies (2) with $h = \frac{2}{3}$, but there is no fixed point for f .

Unexpectedly, Proposition 1 does extend to *continuous* maps satisfying (2) even if (X, d) is unbounded and d is not symmetric. Such a space (X, d) endowed with the right convergence operator we also call an E -space.

THEOREM 2. *Let X be a nonempty set and $d : X \times X \mapsto \mathbf{R}_+$ be a function such that, given $x, y \in X$,*

$$d(x, y) = 0 \quad \text{iff} \quad x = y.$$

Let f be a selfmap on X such that condition (2) holds and f is d -continuous, i.e., given $\{x_n\}_{n=1}^\infty$ and x in X , $d(x_n, x) \rightarrow 0$ implies $d(fx_n, fx) \rightarrow 0$. If the E -space (X, d) is complete and d satisfies Wilson's Axiom III [17], i.e., given $\{x_n\}_{n=1}^\infty$, x and y in X ,

$$d(x_n, x) \rightarrow 0 \quad \text{and} \quad d(x_n, y) \rightarrow 0 \quad \text{imply that} \quad x = y,$$

then f has a unique fixed point p , and $d(f^n x, p) \rightarrow 0$, for all $x \in X$.

Proof. Define $\alpha(x) := d(x, fx)$, for $x \in X$. Then (2) easily implies that $\alpha^{-1}(0)$ is at most a singleton and $\alpha(fx) \leq h\alpha(x)$, for $x \in X$. Hence if

$$\bar{d}(x, y) := \max\{\alpha(x), \alpha(y)\} \quad \text{for } x \neq y, \quad \text{and} \quad \bar{d}(x, x) := 0 \quad \text{for } x \in X,$$

then one can verify that \bar{d} is the metric; in particular, $\bar{d}(x, y) \leq \max\{\bar{d}(x, z), \bar{d}(z, y)\}$, for $x, y, z \in X$, so \bar{d} is the ultrametric (see, e.g., [7], p.504). Moreover, f is the Banach contraction with respect to \bar{d} with the same constant h as in (2). By the proof of the Contraction Principle, for any $x \in X$ the sequence $\{f^n x\}_{n=1}^\infty$ is \bar{d} -Cauchy. By (2), for $n, m \in \mathbf{N}$, if $f^n x \neq f^m x$ then

$$d(f^{n+1}x, f^{m+1}x) \leq h \max\{\alpha(f^n x), \alpha(f^m x)\} = h\bar{d}(f^n x, f^m x).$$

Hence, $d(f^{n+1}x, f^{m+1}x) \leq h\bar{d}(f^n x, f^m x)$, which holds also in case, in which $f^n x = f^m x$. Therefore, we may conclude that $\{f^n x\}_{n=1}^\infty$ is \bar{d} -Cauchy. By the completeness, there is a $p \in X$ such that $d(f^n x, p) \rightarrow 0$. Then $d(f^{n+1}x, fp) \rightarrow 0$ because of the continuity of f . Hence, $p = fp$ since d satisfies Axiom III. Moreover, p does not depend on x , since the fixed point is unique. \square

REMARK 3. If d is continuous with respect to the first variable, i.e., given $\{x_n\}_{n=1}^\infty$, x, y in X , $d(x_n, x) \rightarrow 0$ implies $d(x_n, y) \rightarrow d(x, y)$ (this forces Axiom III for such a d), then the assumption in Theorem 2 that f be continuous can be dropped. To see that, observe that, by the proof of Theorem 2, given $x \in X$, there is a $p \in X$ such that $d(f^n x, p) \rightarrow 0$. By the continuity of d and the inequality

$$d(f^{n+1}x, fp) \leq h \max\{d(f^n x, f^{n+1}x), d(p, fp)\}, \quad \text{for } x \in X,$$

we get letting $n \rightarrow \infty$ that $d(p, fp) \leq hd(p, fp)$, and hence $p = fp$.

REMARK 4. Theorem 2 can be extended to maps satisfying more general contractive condition:

$$d(fx, fy) \leq \phi(\max\{d(x, fx), d(y, fy)\}), \quad \text{for } x, y \in X,$$

where ϕ is a function as in Theorem 1. Then the above given proof needs a slight modification only; that $\{f^n x\}_{n=1}^\infty$ is \bar{d} -Cauchy follows this time from the proof of Theorem 1.2 [13] and the fact that f satisfies (4) in a metric space (X, \bar{d}) .

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J. JACHYMSKI

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
UL. ŻWIRKI 36
90-924 ŁÓDŹ, POLAND
JACHYMSK@LODZ1.P.LODZ.PL

J. MATKOWSKI

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
UL. WILLOWA 2
43-300 BIELSKO-BIALA, POLAND