Remark on Generalization of Minkowski's Inequality

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Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) \leq 1$. We give some general conditions for a bijection $\varphi : [0, \infty) \mapsto [0, \infty)$, such that

$$\varphi^{-1}\left(\int_\Omega \varphi \circ |x + y| \, d\mu\right) \leq \varphi^{-1}\left(\int_\Omega \varphi \circ |x| \, d\mu\right) + \varphi^{-1}\left(\int_\Omega \varphi \circ |y| \, d\mu\right)$$

for all μ-integrable simple functions $x, y : \Omega \mapsto \mathbf{R}$. This generalizes result from [1].

1. Introduction

For a measure space (Ω, Σ, μ) such that $\mu(\Omega) < \infty$, denote by $S(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable step functions $x : \Omega \mapsto \mathbf{R}_+ (:= [0, \infty))$. Let $\varphi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be an arbitrary bijection. Then the functional $P_\varphi : S(\Omega, \Sigma, \mu) \mapsto \mathbf{R}_+$ given by

$$P_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x| d\mu\right), \qquad x \in S(\Omega, \Sigma, \mu),$$

is well defined. For $\varphi(t) = \varphi(1)t^p$ $(t \geq 0)$ with $p \geq 1$, the functional P_φ coincides with the $\mathscr{L}^p$-norm. In this note we prove the following generalization of Minkowski's inequality:

Theorem. *Let* (Ω, Σ, μ) *be a measure space such that* $\mu(\Omega) \leq 1$. *Suppose* $\varphi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ *satisfies the following conditions:*

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1^0. *φ is bijective, increasing, and differentiable;*
2^0. *φ' is strictly increasing, and locally absolutely continuous;*
3^0. *there exists a superadditive function $g : \mathbf{R}_+ \mapsto \mathbf{R}_+$ such that*

$$g = \frac{\varphi'}{\varphi''} \text{ a.e. } \quad \textit{in } \mathbf{R}_+ .$$

Then for all $x, y \in S(\Omega, \Sigma, \mu)$,

$$P_\varphi(x + y) \leq P_\varphi(x) + P_\varphi(y).$$

This generalizes a result from paper [1] of the second named author where φ is assumed to be of the class $\mathscr{C}^2$ and such that $\varphi'' > 0$ and $\frac{\varphi'}{\varphi''}$ is superadditive in $(0, \infty)$. At the end of this paper we explain the assumption that $\mu(\Omega) \leq 1$.

2. Auxiliary lemma and the proof of Theorem

The proof of the theorem is based on the following.

Lemma. *If $\varphi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ satisfies the conditions $1^0, 2^0, 3^0$ of the theorem, then there exists a sequence of functions $\varphi_n : \mathbf{R}_+ \mapsto \mathbf{R}_+$ such that:*

a) for every $n \in \mathbf{N}$, φ_n is bijective and of the class $\mathscr{C}^\infty$;

b) for every $n \in \mathbf{N}$, $\varphi_n' > 0$, $\varphi_n'' > 0$ in $(0, \infty)$, and the function $\frac{\varphi_n'}{\varphi_n''}$ is superadditive in $(0, \infty)$;

c) for every $a > 0$,

$$\lim_{n \to \infty} \varphi_n = \varphi, \qquad \lim_{n \to \infty} \varphi_n' = \varphi', \quad \text{uniformly on } [0, a];$$

d)

$$\lim_{n \to \infty} \frac{\varphi_n'}{\varphi_n''} = g \quad \text{a.e.} \quad \textit{in} \quad \mathbf{R}_+ \ (\text{and } \textit{in } \mathscr{L}^1_{loc})$$

where g is defined in the theorem; this convergence is uniform on every compact interval of the continuity of g contained in $(0, \infty)$.

Proof. By 1^0 and 2^0 the function $\log \circ \varphi'$ is locally absolutely continuous. Consequently it is equal to a primitive of its derivative

$$(\log \circ \varphi')' = \frac{\varphi''}{\varphi'} = \frac{1}{g}. \tag{1}$$

Take a sequence $\varrho_n : \mathbf{R} \mapsto \mathbf{R}_+$ of $\mathscr{C}^\infty$-smooth even functions such that

$$\operatorname{supp} \varrho_n \subset \left[-\frac{1}{n}, \frac{1}{n} \right], \quad \int_{-\infty}^{+\infty} \varrho_n = 1, \tag{2}$$

and define $g_n : \mathbf{R}_+ \mapsto \mathbf{R}_+$ by the formula

$$g_n(t) = \int_0^\infty g(ts)\,\varrho_n(1-s)\,ds, \qquad t \geq 0, \quad n \in \mathbf{N}.$$

Note that g_n is increasing, bijective, superadditive, of the class $\mathscr{C}^\infty$, and

$$\lim_{n\to\infty} g_n = g \quad \text{a.e.} \quad in\ \mathbf{R}_+.$$

Since g is increasing, we have

$$(3) \qquad g_n(t) \geq \int_1^\infty g(ts)\,\varrho_n(1-s)\,ds \geq \int_1^\infty g(t)\,\varrho_n(1-s)\,ds = \frac{g(t)}{2}$$

for all $t \geq 0$.

Now we are going to define φ_n, $n \in \mathbf{N}$. First we define its derivative φ_n' in such a way that $\log \circ \varphi_n'$ is the primitive of $\frac{1}{g_n}$ for which $\varphi_n'(1) = \varphi'(1)$. The value $\varphi_n'(0)$ is well-defined if $\int_0 \frac{1}{g_n} < \infty$; otherwise we put $\varphi_n'(0) = 0$. By (1), (3) and the Lebesgue majorization theorem, we have

$$(4) \qquad \lim_{n\to\infty} \varphi_n' = \varphi'$$

pointwise on $(0, \infty)$. As all functions here are continuous and increasing, it follows that the convergence (4) is uniform on every compact interval contained in $(0, \infty)$. For proving that (4) holds uniformly on $[0, 1]$ too, we will distinguish two cases depending on $\varphi'(0) > 0$ or $\varphi'(0) = 0$.

If $\varphi'(0) > 0$, then by (1) the function $\frac{1}{g}$ is integrable on $[0, 1]$, and using the Lebesgue majorization theorem, as above, we obtain that (4) holds pointwise, and, therefore, uniformly on $[0, 1]$.

Now suppose that $\varphi'(0) = 0$. We know that φ' is continuous, increasing, (4) holds uniformly on $[\varepsilon, 1]$ for every $\varepsilon \in (0, 1)$, and that φ_n' is increasing and positive on $(0, 1]$. Thus the convergence must be uniform on $[0, 1]$, too.

The definition of the function φ_n, for which $\varphi_n(0) = 0$, is obvious. Evidently, $\lim_{n\to\infty} \varphi_n = \varphi$ uniformly on $[0, a]$ for every $a > 0$, and the lemma is proved.

Now we give the

Proof of theorem. Let φ_n, $n \in \mathbf{N}$, be the sequence of functions constructed in the lemma, and let $x, y \in S(\Omega, \Sigma, \mu)$ be arbitrary. Then by Theorem 3 in [1] we have

$$\varphi_n^{-1}\left(\int_\Omega \varphi_n \circ |x+y|\,d\mu\right) \leq \varphi_n^{-1}\left(\int_\Omega \varphi_n \circ |x|\,d\mu\right) + \varphi_n^{-1}\left(\int_\Omega \varphi_n \circ |y|\,d\mu\right).$$

Letting $n \to \infty$ here and making use of the lemma, we get

$$\varphi^{-1}\left(\int_\Omega \varphi \circ |x+y|\,d\mu\right) \leq \varphi^{-1}\left(\int_\Omega \varphi \circ |x|\,d\mu\right) + \varphi^{-1}\left(\int_\Omega \varphi \circ |y|\,d\mu\right),$$

which, by the definition of P_φ, completes the proof.

Remark 1. Suppose that (Ω, Σ, μ) is a measure space such that there exist $A, B \in \Sigma$ satisfying the condition

$$0 < \mu(A) < 1 < \mu(B) < \infty .$$

In [1] it is shown that if $\varphi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is bijective, φ^{-1} continuous at 0, and

$$P_\varphi(x + y) \leq P_\varphi(x) + P_\varphi(y) \qquad \text{holds for all } x, y \in S(\Omega, \Sigma, \mu),$$

then $\varphi(t) = \varphi(1)t^p$ $(t \geq 0)$, for some $p \geq 1$. This shows in particular that the assumption $\mu(\Omega) \leq 1$ is essential.

In this connection let us also mention the following

Remark 2. Suppose that (Ω, Σ, μ) has the following property: for every $A \in \Sigma$

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) \geq 1 .$$

Under this assumption it is proved in [2] that if $\varphi: \mathbf{R}_+ \mapsto \mathbf{R}_+$ is a convex homeomorphism of $\mathbf{R}_+$ such that φ is geometrically convex in $(0, \infty)$, i.e. that

$$\varphi\left(\sqrt{st}\right) \leq \sqrt{\varphi(s)\,\varphi(t)} \qquad \text{for all} \quad s, t > 0 ,$$

then

$$P_\varphi(x + y) \leq P_\varphi(x) + P_\varphi(y) \qquad \text{for all} \quad x, y \in S(\Omega, \Sigma, \mu),$$

In the proof of this result the one-sided derivatives and Zygmund's lemma are used. It turns out that the argument can be simplified if we work with smooth functions φ. The following result permits us to do it.

Proposition. *Suppose that φ is a convex and geometrically convex homeomorphism of $\mathbf{R}_+$ onto itself. Then there exists a sequence φ_n, $n \in \mathbf{N}$, of $\mathscr{C}^\infty$-smooth convex and geometrically convex diffeomorphisms of $\mathbf{R}_+$ onto itself such that*

$$\lim_{n \to \infty} \varphi_n = \varphi$$

uniformly on $[0, a]$ for every $a > 0$.

Proof. Taking the function ϱ_n given by (2) in the previous proof, we define φ_n as follows

$$\varphi_n(t) := \exp \int \varrho_n(u) \log \varphi(t\, e^{-u})\, du , \quad t > 0,$$

and $\varphi_n(0) = 0$ to have φ_n continuous at 0. Since $\{\varphi_n\}$ converges to φ pointwise on $\mathbf{R}_+$, the monotonicity of φ_n and φ implies that the convergence is uniform on $[0, a]$ for every $a > 0$.

Now we have for all $s, t > 0$

$$\varphi_n(\sqrt{st}) = \exp \int \varrho_n(u) \log \varphi\,(\sqrt{st}\, e^{-u})\, du \le \exp \int \varrho_n(u) \log \sqrt{\varphi(se^{-u})\varphi(te^{-u})}\, du =$$

$$\exp \int \varrho_n(u) \left[\frac{1}{2}(\log \varphi(se^{-u}) + \log \varphi(te^{-u}))\right] du = \sqrt{\varphi_n(s)\,\varphi_n(t)}$$

which shows that φ_n is geometrically convex.

Now we shall show that φ_n is convex. As φ is convex with $\varphi(0) = 0$, the function $\frac{\varphi(t)}{t}$ is increasing, too. For $0 < s < t$ we have

$$\varphi_n(s) = \exp \int \varrho_n(u) \log \varphi(s\, e^{-u})\, du \le \exp \int \varrho_n(u) \log \frac{s}{t}\, \varphi(te^{-u}) =$$

$$\exp \int \varrho_n(u) \left[\log \frac{s}{t} + \log \varphi(te^{-u})\right] du = \frac{s}{t}\, \varphi_n(t),$$

which was to be shown.

For showing that φ_n is convex, we use the following known property of geometrically convex functions φ: if the function $\frac{\varphi_n(t)}{t}$ is increasing, then φ_n is convex. Let us show it briefly. Suppose that φ_n is not convex; then there are points $0 < s < u < t$ and a linear function l such that

$$(5) \qquad \varphi_n(s) - l(s) = \varphi_n(t) - l(t) = 0 \quad \text{and} \quad \varphi_n(u) - l(u) > 0\,.$$

The points s, t can be changed without changing l so that (5) holds for all $u \in (s, t)$. For $u = \sqrt{st}$ we get from (5) by a simple calculation

$$\varphi_n(\sqrt{st}) > \varphi_n(s)\, \frac{\sqrt{t}}{\sqrt{s} + \sqrt{t}} + \varphi_n(t)\, \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t}}\,.$$

Thanks to the geometrical convexity of φ_n, it follows

$$(\sqrt{s} + \sqrt{t})\, \sqrt{\varphi_n(s)\,\varphi_n(t)} > \varphi_n(s)\sqrt{t} + \varphi_n(t)\sqrt{t}\,,$$

$$(\sqrt{s} + \sqrt{t})\, \sqrt{\frac{\varphi_n(s)\,\varphi_n(t)}{st}} > \frac{\varphi_n(s)}{s}\, \sqrt{s} + \frac{\varphi_n(t)}{t}\, \sqrt{t}\,,$$

$$\sqrt{\frac{\varphi_n(s)}{s}}\, \sqrt{s}\left(\sqrt{\frac{\varphi_n(t)}{t}} - \sqrt{\frac{\varphi_n(s)}{s}}\right) > \sqrt{\varphi_n(t)t}\, \sqrt{t}\left(\sqrt{\frac{\varphi_n(t)}{t}} - \sqrt{\frac{\varphi_n(s)}{s}}\right).$$

We see that the inequality $\sqrt{\frac{\varphi_n(t)}{t}} + \sqrt{\frac{\varphi_n(s)}{s}} \geqq 0$ is not possible, so the function $\frac{\varphi_n(t)}{t}$ could not be increasing if φ_n were not convex. the proposition is proved.

References

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