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On stability of the homogeneity condition

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Dedicated with affection to Professor János Aczél on the occasion of his seventieth birthday

Abstract. Let f be a function defined on a cone S with the values in a sequentially complete locally convex linear topological Hausdorff space Y. If there exist a bounded subset V of Y and an open interval $(a,b) \subset (1,\infty)$ such that for all $x \in S$ and every $\lambda \in (a,b)$ the condition $\lambda^{-1}f(\lambda x) - f(x) \in V$ holds, then there exists a unique positively homogeneous mapping $F : S \to Y$ such that the difference $F(x) - f(x)$ is uniformly bounded on S.

Introduction

In a recent paper [3] J. Tabor proved that every mapping $f : X \to Y$ from a real vector space X into a normed space Y satisfying the inequality

$$\|\alpha^{-1}f(\alpha x) - f(x)\| \leq \varepsilon \tag{1}$$

for all $\alpha \in \mathbb{R}$ and $x \in X$, where $\varepsilon \geq 0$ is given, must be homogeneous. In the next paper [4] written jointly with J. Tabor, Jr., they generalized this result which is interpreted as a superstability of the homogeneity condition. The same assertion holds true if we assume that condition (1) is fulfilled for every $x \in X$ and $\alpha \in (-\delta,\delta) \setminus \{0\}$, where $\delta > 0$ is a constant. In fact, setting $y = \alpha x$ in (1) we can easily show that the analogous inequality with $\tilde{\varepsilon} = max\{\delta\varepsilon, \varepsilon\}$ on the right hand side is fulfilled for every $x \in X$ and $\alpha \in (-\infty, -\frac{1}{\delta}) \cup (-\delta,\delta) \setminus \{0\} \cup (\frac{1}{\delta}, \infty)$. Now for a fixed $\beta \in (\frac{1}{\delta^2}, \infty)$, every $\gamma, |\gamma| \leq \frac{1}{\delta}$, may be written in the form $\gamma = \alpha\beta$ with an $\alpha \in (-\delta,\delta) \setminus \{0\}$. Hence

$$\begin{aligned}\|\gamma^{-1}f(\gamma x) - f(x)\| \leq{} & \|\alpha^{-1}\beta^{-1}f(\alpha\beta x) - \beta^{-1}f(\beta x)\| + \|\beta^{-1}f(\beta x) - f(x)\| \leq \\ \leq{} & \beta^{-1}\varepsilon + \tilde{\varepsilon} \leq (\delta^2 + \delta)\varepsilon,\end{aligned}$$

which by Tabor's result implies that f is homogeneous function.

Note also that if we assume condition (1) for every ($x \in X$ and) $\alpha \in \mathbb{R}, |\alpha| > \delta$, where $\delta > 0$ is a constant, then Tabor's assertion does not hold. To see this, it is enough to consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$f(x) = \begin{cases} 0 & \text{for } |x| \geq 1 \text{ or } x = 0 \\ 1 & \text{for } x \in (0,1) \\ -1 & \text{for } x \in (-1,1). \end{cases}$$

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Evidently f is not a homogeneous function. On the other hand it is not hard to check that for all α, $|\alpha\,| \geq 1$ and $x \in \mathbb{R}$ we have

$$|\alpha^{-1} f(\alpha x) - f(x)| \leq 2$$

The condition (1) implies the following two inequalities

$$\|\alpha^{-1} f(\alpha x) - f(x)\| \leq \varepsilon \quad \text{for every } \alpha > 0 \text{ and } x \in X \tag{2}$$

and

$$\|f(x) + f(-x)\| \leq \varepsilon \quad \text{for every } x \in X. \tag{3}$$

Conversely, conditions (2) and (3) imply the condition (1) (with 2ε instead of ε on the right hand side). In fact, for any $\alpha < 0$, by (2) and (3) we have

$$\begin{aligned}\|\alpha^{-1} f(\alpha x) - f(x)\| &\leq \| - \alpha^{-1} f((-\alpha)(-x)) - f(-x)\| + \\ &+ \|f(-x) + f(x)\| \leq 2\varepsilon.\end{aligned}$$

Thus Tabor's result says that every $f : X \to Y$ satisfying conditions (2) and (3) is homogeneous. An example of the function $f(x) = |x|$, $x \in \mathbb{R}$, shows that the condition (3) is essential here.

In the present paper we deal with the stability of the homogeneity condition. We prove that for every function f satisfying the condition

$$\alpha^{-1} f(\alpha x) - f(x) \in V, \qquad \alpha \in A, \;\; x \in S,$$

where S is a cone in X , $A \subset (1, \infty)$ is a set with a nonempty interior, and V is a bounded subset of Y, there exists a unique positively homogeneous function $F : S \to Y$ such that the difference $F(x) - f(x)$ is uniformly bounded. The form of F is also given. The continuity of f permits us to replace the assumption int A $\neq \emptyset$ by the condition that A contains two noncommensurable numbers.

Moreover, a suitable result in which the difference $f(x + y) - f(x) - f(y)$ is assumed to be uniformly bounded, is also given.

1. Auxiliary results

In the sequel the letters $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{R}_+$ stand for positive integers, integers, rationals, reals and nonnegative reals, respectively. Throughout this paper the symbol X stands for a real linear space and Y for a sequentially complete locally convex linear topological Hausdorff space. By seq cl V we will denote the sequential closure of V, and by conv V the convex hull of V. A set $S \subset X$ is said to be a cone iff $tS \subset S$, for all $t > 0$. A cone S such that $S + S \subset S$ is said to be convex.

Lemma 1. *Let f be a function defined on a cone S and with the values in Y. If there exist $A \subset (1, \infty), A \neq \emptyset$, and a bounded $V \subset Y$ such that*

$$\alpha^{-1} f(\alpha x) - f(x) \in V, \qquad \alpha \in A, x \in S, \tag{4}$$

then for every $\alpha \in A$:

$$1^\circ \left\{ \begin{array}{l} \text{the function } F_\alpha : S \to Y \text{ given by} \\ F_\alpha(x) := \lim_{n\to\infty} \alpha^{-n} f(\alpha^n x), \quad x \in S, \end{array} \right.$$

is well defined, and the convergence is uniform on S;

$$2^\circ \left\{ \begin{array}{l} F_\alpha \text{ is } \alpha\text{-homogeneous, i.e. } F_\alpha(\alpha x) = \alpha F_\alpha(x), \ x \in S, \text{ and} \\ F_\alpha(x) - f(x) \in \alpha(\alpha-1)^{-1} \mathrm{seq\ cl\ conv}\,(V \cup \{0\}) \ ; \end{array} \right.$$

$$3^\circ \left\{ \begin{array}{l} \text{for all } \beta \in A \\ F_\beta(x) = F_\alpha(x), \quad x \in S, \end{array} \right.$$

i.e. there exists a unique $F : S \to Y$ such that $F_\beta = F$ for all $\beta \in A$.

Proof. Let us fix an arbitrary $\alpha \in A$. For all $n, m \in \mathbb{N}$ and $x \in S$ we have

$$\alpha^{-n-m} f(\alpha^{n+m}x) - \alpha^{-m} f(\alpha^m x) = \alpha^{-m}[\alpha^{-n} f(\alpha^{n+m}x) - f(\alpha^m x)] =$$

$$= \alpha^{-m} \sum_{k=1}^{n} \alpha^{-(k-1)} [\alpha^{-1} f(\alpha\alpha^{m+k-1}x) - f(\alpha^{m+k-1}x)]$$

$$\in \alpha^{-m} \sum_{k=1}^{n} \alpha^{-(k-1)} V \subset \alpha^{-(m-1)}(\alpha-1)^{-1} \mathrm{conv}(V \cup \{0\})$$

which shows that $(\alpha^{-n} f(\alpha^n x))$ is a uniformly convergent Cauchy sequence. It follows that the function $F_\alpha : S \to Y$ given in 1° is well defined and we have

$$F_\alpha(\alpha x) = \lim_{n\to\infty} \alpha^{-n} f(\alpha^{n+1}x) = \alpha \lim_{n\to\infty} \alpha^{-(n+1)} f(\alpha^{n+1}x) = \alpha F_\alpha(x).$$

The identity

$$\alpha^{-n} f(\alpha^n x) - f(x) = \frac{\alpha}{\alpha-1} \left(\sum_{k=1}^{n} \frac{\alpha-1}{\alpha^k} [\alpha^{-1} f(\alpha\alpha^{k-1}x) - f(\alpha^{k-1}x)] + \alpha^{-n} 0 \right)$$

and the condition (4) imply

$$F_\alpha(x) - f(x) \in \frac{\alpha}{\alpha-1} \mathrm{seq\ cl}(\mathrm{conv}\ (V \cup \{0\})) \ , \qquad x \in S,$$

which proves 2°. Hence, for an arbitrary fixed $\alpha, \beta \in A$ and all $x \in S$

$$\beta^{-n} F_\alpha(\beta^n x) - \beta^{-n} f(\beta^n x) \in \beta^{-n} \alpha(\alpha-1)^{-1} \mathrm{seq\ cl\ conv}(V \cup \{0\}).$$

Making use of 1°, we get

$$\lim_{n\to\infty} \beta^{-n} F_\alpha(\beta^n x) = F_\beta(x), \qquad x \in S,$$

and, consequently,

$$F_\beta(x) - F_\alpha(x) = \lim_{n\to\infty} [\beta^{-n} F_\alpha(\beta^n x) - \alpha^{-n} F_\beta(\alpha^n x)] =$$

$$= \lim_{n\to\infty} \alpha^{-n}\beta^{-n} [\alpha^n F_\alpha(\beta^n x) - \beta^n F_\beta(\alpha^n x)].$$

By virtue of 2° we obtain

$$F_\beta(x) - F_\alpha(x) = \lim_{n\to\infty} \alpha^{-n}\beta^{-n}[F_\alpha(\alpha^n\beta^n(x) - f(\alpha^n\beta^n x) + $$
$$+ f(\alpha^n\beta^n x) - F_\beta(\alpha^n\beta^n x)] = 0$$

which proves 3°. The proof of Lemma 1 is complete.

Remark 1. An analogous Lemma holds true if the condition $A \subset (1,\infty)$ is replaced by $A \subset (0,1)$, and the basic relation (4) by the following one

$$\alpha f(\alpha^{-1}x) - f(x) \in V.$$

Lemma 2. *Let $S \subset X$ be a cone, and $f, F_1, F_2 : S \to Y$ mappings. If F_1, F_2 are positively homogeneous and there exist a function $g : S \to \mathbb{R}_+$ and a bounded subset V of Y such that*

$$F_i(x) - f(x) \in g(x)V, \qquad x \in S, \qquad i = 1,2;$$

and

$$\inf\left\{\tfrac{g(tx)}{t};\ t > 0\right\} = 0, \qquad x \in S,$$

then $F_1 = F_2$.

Proof. We have

$$F_1(x) - F_2(x) = t^{-1}(F_1(tx) - F_2(tx)) = t^{-1}(F_1(tx) - f(tx) + f(tx) - F_2(tx)) \in$$
$$\in t^{-1}(g(tx)V - g(tx)V) \subset \tfrac{g(tx)}{t}V - \tfrac{g(tx)}{t}V$$

for all $x \in S$ and $t > 0$. According to our assumptions we get $F_1(x) = F_2(x)$ for all $x \in S$, which was to be shown.

2. Stability of the homogeneity condition

We begin this section with the following

Theorem 1. *Let $S \subset X$ be a cone and $f : S \to Y$ a mapping. If there exist $A \subset (1,\infty)$, $\mathrm{int}A \neq \emptyset$, and a bounded set $V \subset Y$ such that*

$$\alpha^{-1}f(\alpha x) - f(x) \in V, \qquad \alpha \in A, \quad x \in S, \tag{5}$$

then there exists a unique positively homogeneous mapping $F : S \to Y$ such that

$$F(x) - f(x) \in c(c-1)^{-1}\mathrm{seq\ cl\ conv}(V \cup \{0\}), \qquad x \in S,$$

where $c := \sup(A)$. In particular, if $\sup(A) = \infty$ then

$$F(x) - f(x) \in \mathrm{seq\ cl\ conv}(V \cup \{0\}), \qquad x \in S.$$

Moreover,

$$F(x) = \lim_{n \to \infty} \alpha^{-n} f(\alpha^n x), \qquad x \in S,$$

and the convergence is uniform on S.

Proof. Let us fix an $\alpha \in A$ and put $F := F_\alpha$. In view of Lemma $1^\circ - 2^\circ$ we have

$$F(\lambda x) = \lambda F(x), \qquad \lambda \in A, \quad x \in S,$$

Replacing x by $\lambda^{-1}x$ we hence get

$$F(\lambda^{-1}x) = \lambda^{-1}F(x), \qquad \lambda \in A, \quad x \in S,$$

and, by induction,

$$F(\lambda_1 \cdots \lambda_n \cdot \mu_1^{-1} \cdots \mu_m^{-1} x) = \lambda_1 \cdots \lambda_n \cdot \mu_1^{-1} \cdots \mu_m^{-1} F(x)$$

for all $n, m \in \mathbb{N}$; $\lambda_1, \ldots \lambda_n$; $\mu_1, \ldots \mu_m \in A, x \in S$. Since int $A \neq \emptyset$, we have

$$\{\lambda_1 \cdots \lambda_n \cdot \mu_1^{-1} \cdots \mu_m^{-1};\ \lambda_1, \ldots, \lambda_n,\ \mu_1, \ldots, \mu_m \in A,\ \ n, m \in \mathbb{N}\} = (0, \infty).$$

Thus

$$F(\lambda x) = \lambda F(x), \qquad \lambda \in (0, \infty), \quad x \in S,$$

which means that F is positively homogeneous.

By the definition of F and by 2° we obtain

$$F(x) - f(x) \in \alpha(\alpha - 1)^{-1} \text{seq cl}(V \cup \{0\}), \qquad \alpha \in A, \quad x \in S.$$

Since the function $A \ni \alpha \to \alpha(\alpha - 1)^{-1}$ is decreasing and the left hand side does not depend on α, this condition holds true for $\alpha = c := \sup(A)$. If $c = +\infty$ then, of course, we get $F(x) - f(x) \in \text{seq cl}(V \cup \{0\})$, $x \in S$. This completes the proof.

Remark 2. It is sufficient to assume the condition (5) *for all* $x \in S$ *and* $\alpha \in A$ *such that the set* $\underbrace{A \cdots A}_{p} := \{\alpha_1 \cdots \alpha_p;\ \ \alpha_i \in A,\ \ i = 1, \ldots, p\}$ *has a nonempty interior, for some* $p \in \mathbb{N}$.

Remark 3. The condition from Remark 2 is fulfilled if the inner Lebesque measure of A is positive.

Remark 4. The assumption int$A \neq \emptyset$ can be replaced by the following weaker one: *there exists a nonempty open interval* $I \subset (1, \infty)$ *such that for every* $\lambda \in I$ *there is an* $\alpha \in A$ *such that* $\lambda\alpha \in A$.

Example. Let $f : [0, \infty) \to \mathbb{R}$ be defined by the following formula

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 3x - 2, & x \in [1, 2) \\ x + 2, & x \in [2, \infty) \end{cases}$$

It is easy to check that F satisfies (5) with $A = \{2\}$ and $V = [-1,1]$. Moreover, the unique positively homogeneous function $F : [0,\infty) \to \mathbb{R}$ lying close to f is the identity $F(x) = x$. We see that

$$sup\{|F(x) - f(x)|;\ \ x \in [0,\infty)\} = 2 = \frac{2}{2-1} \cdot 1.$$

This shows that the relevant estimation obtained in the assertion of our Theorem 1 is the best one.

Proposition. *Let $\alpha, \beta > 1$ be such that $\log \alpha$ and $\log \beta$ are not commensurable. Suppose that $V \subset Y$ is a bounded subset of Y. If $f : (0,\infty) \to Y$ is continuous at least at one point and satisfies the condition*

$$\alpha^{-1} f(\alpha t) - f(t),\ \ \beta^{-1} f(\beta t) - f(t) \in V, \qquad t > 0,$$

then there exists a unique positively homogeneous function $\varphi : (0,\infty) \to Y$ such that

$$\varphi(t) - f(t) \in c(c-1)^{-1} \text{seq cl conv}(V \cup \{0\}), \qquad t > 0 \tag{6}$$

where $c := \max\{\alpha, \beta\}$. Moreover

$$\varphi(t) = \lim_{n\to\infty} \alpha^{-n} f(\alpha^n t) = \lim_{n\to\infty} \beta^{-n} f(\beta^n t), \qquad t > 0, \tag{7}$$

and the convergence is uniform on $(0,\infty)$.

Proof. By the Kronecker theorem the set

$$D := \{\alpha^n \beta^m;\ n, m \in \mathbb{Z}\}$$

is dense in $(0,\infty)$.

In view of Lemma 1° - 3° (here $A = \{\alpha, \beta\}$) the function $\varphi : (0,\infty) \to Y$ defined by the formula (7) satisfies the functional equations

$$\varphi(\alpha t) = \alpha\varphi(t), \qquad \varphi(\beta t) = \beta\varphi(t), \qquad t > 0,$$

and, consequently,

$$\varphi(\lambda t) = \lambda\varphi(t),\ \ \lambda \in D, \qquad t > 0. \tag{8}$$

Because the convergence in (7) is also uniform (1° of Lemma 1) on $(0,\infty)$ our assumption of f implies the continuity of φ at least at one point, say $t_\circ > 0$. Take arbitrary $t > 0$ and a sequence $\lambda_n \in D$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \lambda_n = t_\circ t^{-1}$. Letting $n \to \infty$ in the relation (comp. (8))

$$\varphi(\lambda_n t) = \lambda_n \varphi(t), \qquad t > 0$$

we get

$$\varphi(t_\circ) = t_\circ t^{-1} \varphi(t)$$

i.e.

$$\varphi(t) = \frac{\varphi(t_\circ)}{t_\circ} t, \qquad t > 0,$$

(cf. also [2]). The condition (6) is a consequence of Lemma 1, and the uniqueness follows from Lemma 2. This completes the proof.

Immediately from this Proposition we obtain

Theorem 2. *Let X be a real linear topological space, $S \subset Y$ a cone, and $f : S \to Y$ a continuous mapping. Suppose that $V \subset Y$ is a bounded subset of Y, and $A \subset (1, \infty)$ contains at least two elements α and β such that $\log \alpha$ and $\log \beta$ are not commensurable. If for all $\alpha \in A$ and $x \in S$*

$$\alpha^{-1} f(\alpha x) - f(x) \in V$$

then there exists a unique positively homogeneous function $F : S \to Y$ such that

$$F(x) - f(x) \in c(c-1)^{-1} \text{seq cl conv}(V \cup \{0\}), \qquad x \in S,$$

where $c = \sup (A)$. In particular, if $\sup (A) = \infty$, then

$$F(x) - f(x) \in \text{seq cl conv}(V \cup \{0\}).$$

Moreover,

$$F(x) = \lim_{n \longrightarrow \infty} \alpha^{-n} f(\alpha^n x), \qquad x \in S,$$

and the convergence is uniform on S.

Remark 5. According to the Proposition, the assumption of continuity of f in Theorem 2 can be replaced by the following weaker one: *for every $x \in S$ the function $(0, \infty) \ni t \to f(tx)$ is continuous at least at one point.*

3. Stability of linear functions

The main result of this section reads as follows:

Theorem 3. *Let $S \subset X$ be a convex cone and $f : S \to Y$ a mapping. If there exist $A \subset (1, \infty)$ such that int $A \neq \emptyset$ and bounded subset V and V_1 of Y such that*

$$f(x+y) - f(x) - f(y) \in V, \qquad x, y \in S$$

and

$$\alpha^{-1} f(\alpha x) - f(x) \in V_1, \qquad \alpha \in A, \quad x \in S,$$

then there exists a unique linear function $a : S \to Y$ such that

$$a(x) - f(x) \in \text{seq cl conv}(V \cup (-V)).$$

Proof. In view of the Gajda theorem (cf. [1]) there exists a unique additive function $a : S \to Y$ such that $a(x) - f(x) \in \text{seq cl conv}(V \cup (-V))$. On the other hand, by

Lemma 1, there exists a positively homogeneous function $F : S \longrightarrow Y$ satisfying the condition

$$F(x) - f(x) \in c(c-1)^{-1}\text{seq cl conv}(V_1 \cup \{0\}).$$

Hence, for any rational $r > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} r^n(a(x) - F(x)) &= a(r^n x) - F(r^n x) = \\ &= a(r^n x) - f(r^n x) + f(r^n x) - F(r^n x) \in \\ &\in \text{seq cl conv}(V \cup (-V)) + c(c-1)^{-1}\text{seq cl conv}(V_1 \cup \{0\}). \end{aligned}$$

and therefore $F(x) = a(x)$ for all $x \in S$. By Lemma 1 the function a is homogeneous and, consequently, linear. This completes the proof.

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