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REMARKS ON MULTIPLICATION OF SUBADDITIVE FUNCTIONS

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This paper is dedicated to the memory of Professor Tadeusz Świątkowski.

Suppose that $g : (0,\infty) \to \mathbf{R}$ is nonnegative. We prove that if for every nonnegative and subadditive function $f : (0, \infty) \to \mathbf{R}$ the product gf is subadditive, then g is decreasing.

INTRODUCTION

It is obvious that the sum of two subadditive functions is subadditive, but an analogous fact fails to hold for the product of subadditive functions. However it is easy to observe that the following result is true (cf. Hille-Phillips [1], p. 245, Theorem 7.6.4): Let $g : (0, \infty) \to \mathbf{R}$ be nonnegative. If g is decreasing, then for every nonnegative and subadditive function $f : (0, \infty) \to \mathbf{R}$, the product gf is subadditive. In this note we prove the converse implication. Moreover, we show that for all positive integers n the n-th powers of g are subadditive if, and only if, $g(x + y) \leqslant \max(g(x), g(y))$ for all $x,y > 0$. Some examples and complementary results are given.

1. PRELIMINARY REMARKS AND A LEMMA

By **Z**, **Q** and $\mathbf{R}_+$ we denote, respectively, the set of integers, rationals, and nonnegative reals.

A function $f : (0, \infty) \to \mathbf{R}$ is said to be *subadditive* if

$$f(x + y) \leqslant f(x) + f(y), \qquad x,y > 0;$$

and *max-subadditive if*

$$f(x + y) \leqslant \max(f(x), f(y)), \qquad x,y > 0.$$

Let us note the following obvious

Remark 1. If $g : (0, \infty) \to \mathbf{R}_+$ is max-subadditive then it is subadditive, and

$$g(kx) \leqslant g(x), \qquad x > 0; k \in \mathbf{N}.$$

For every function $f_o : (0,1] \to \mathbf{R}$ there exists a unique function $f : \mathbf{R} \to \mathbf{R}$ such that $f|_{(0,1]} = f_o$ and

$$f(x + 1) = f(x), \qquad x \in \mathbf{R}.$$

The function f is said to be the *1-periodic extension* of f_o on **R**. In the same way we define the periodic extension of f_o on $(0, \infty)$.

We need the following easy to verify

Lemma. *For a fixed* $r \in (0,1)$ *denote by* $h : [0,1] \to \mathbf{R}_+$ *the function given by*

$$h(x) := \begin{cases} x & , \quad x \in [0,r] \\ \dfrac{-r}{1-r}(x - r) + r, & x \in (r,1] \end{cases}$$

Let $f : \mathbf{R} \to \mathbf{R}$ *be the periodic extension of* h. *Then* f *is nonnegative and subadditive in* **R**, *and*

$$f(k) = 0, \qquad f(k + r) = r, \qquad k \in \mathbf{Z}.$$

In [3] a more general result is presented.

Remark 2. Obviously, in the above lemma, the 1-periodic function can be replaced by p-periodic, where p is an arbitrary positive number. Moreover, multiplying the function h by the positive number r^{-1}, we can have $h(r) = 1$. It follows that *for every* $p,x > 0$, *such that* $kp \neq x$ *for all* $k \in \mathbf{N}$, *there exists a p-periodic subadditive function* $f:(0, \infty) \to \mathbf{R}_+$ *such that* $f(x) = 1$ *and* $f(p) = 0$.

2. THE MAIN RESULTS

We begin with the following

Theorem 1. *Suppose that* $g:(0, \infty) \to \mathbf{R}_+$ *is subadditive. Then for every* $n \in \mathbf{N}$, *the function* g^n *(the* n*-th power of* g*) is subadditive if, and only if,* g *is max-subadditive in* $(0, \infty)$.

Proof. Suppose that g^n is subadditive for every $n \in \mathbf{N}$, i.e.

$$(g(x + y))^n \leqslant (g(x))^n + (g(y))^n, \qquad x,y > 0, \qquad n \in \mathbf{N}.$$

Writing this inequality in the form

$$g(x + y) \leqslant [(g(x))^n + (g(y))^n]^{1/n}, \qquad x,y > 0, \qquad n \in \mathbf{N},$$

and letting $n \to \infty$, gives

$$g(x + y) \leqslant \max(g(x), g(y)), \qquad x,y > 0,$$

i.e., g is max-subadditive.

The converse implication is obvious.

Remark 3. The above result can be strengthened in the following way: *If there is a sequence* $p_n > 0$ *such that the functions* g^{p_n}, $n \in \mathbf{N}$, *are subadditive, and*

$$\lim_{n \to \infty} p_n = \infty,$$

then g is max-subadditive.

Remark 4. Clearly, every decreasing function $g\colon (0, \infty) \to \mathbf{R}$ is max-subadditive. It is also easy to check that if a function $g\colon (0, \infty) \to \mathbf{R}_+$ satisfies the inequality

$$g(x+y)(x+y) \leqslant g(x)x + g(y)y, \qquad x,y > 0,$$

then it is max-subadditive in $(0, \infty)$. The examples below show that the converse implication is not true.

Example. If $g\colon (0, \infty) \to \mathbf{R}$ has one of the follwong forms:

$$g(x) := \begin{cases} 0, & x \in \mathbf{N} \\ 1, & x \notin \mathbf{N} \end{cases},$$

$$g(x) := \begin{cases} 0, & x \in \mathbf{Q}, \ x > 0 \\ 1, & x \notin \mathbf{Q}, \ x > 0 \end{cases},$$

$$g(x) := \begin{cases} c, & x \in \mathbf{Q}, \ x > 0 \\ h(x) & x \notin \mathbf{Q}, \ x > 0 \end{cases},$$

where $h\colon (0, \infty) \to \mathbf{R}_+$ is decreasing and $c := \inf\limits_{x>0} h(x)$, then g is max-subadditive.

Thus, in general, the function satisfying the assumption of Theorem 1 (the max-subadditive function) can be quite irregular. It turns out however that if we replace the assumption: „for every positive integer n, the n-th power of g is subadditive", by a suitable stronger one, the function g must be decreasing.

Theorem 2. *Suppose that* $g\colon (0, \infty) \to \mathbf{R}_+$. *Then, for every subadditive function* $f\colon (0, \infty) \to \mathbf{R}_+$, *the product function* $g \cdot f$ *is subadditive if, and only if,* g *is decreasing on* $(0, \infty)$.

Proof. Suppose that

$$g(x+y)\, f(x+y) \leqslant g(x)\, f(x) + g(y)\, f(y), \qquad x,y > 0, \tag{1}$$

for every subadditive $f\colon (0, \infty) \to \mathbf{R}_+$. First we shall show the following

Claim. *Let $x,y > 0$, such that $0 < y < x$, be arbitrary fixed. If*

$$y \neq \frac{x}{k}, \qquad k \in \mathbf{N},$$

then $g(x + y) \leqslant g(x)$.

For an indirect argument suppose that there exist $x > 0$, $y > 0$, such that $ky \neq x$ for every $k \in \mathbf{N}$, and

$$g(x + y) > g(x)\,. \tag{2}$$

In view of Lemma (cf. also Remark 2) there exists a nonnegative periodic subadditive function $f\colon (0, \infty) \to \mathbf{R}_+$ of a period y such that $f(y) = 0$, $f(x) = 1$, and, consequently, $f(x + y) = f(x) = 1$. Hence, making use of (2), we get

$$g(x + y)\, f(x + y) = g(x + y) > g(x) = g(x)\, f(x) = g(x)\, f(x) + g(y)\, f(y)\,,$$

which is a contradiction.

To show that g is decreasing, take arbitrary $x > 0$ and $y > 0$ such that $0 < y < x$, and consider the following two cases:

1° for every $k \in \mathbf{N}$, $ky \neq x$;

2° there is a $k \in \mathbf{N}$ such that $ky = x$.

In the first case, according to our claim, we have $g(x) \geqslant g(x + y)$.

In the second case let us choose a point $z \in (0,y)$ such that z/y is irrational. Applying the claim with $x = ky$, and z instead of y, gives

$$g(x) \geqslant g(x + z)\,.$$

Applying again the claim with $x + z = ky + z$ instead of x, and y replaced by $y - z$, gives

$$g(x + z) \geqq g((x + z) + (y - z)) = g(x + y).$$

Consequently, $g(x) \geqslant g(x + y)$.

Thus we have shown that g is decreasing. Since the reversed implication is obvious (and known), the proof is completed.

3. SOME COMPLEMENTARY RESULTS

In the main result (Theorem 2) we assume that the functions g and f are nonnegative. A partial explanation gives the following

Proposition. *Suppose that* $g\colon (0, \infty) \to \mathbf{R}$. *If for every subadditive function* $f\colon (0, \infty) \to \mathbf{R}$, *the product* $g \cdot f$ *is subadditive in* $(0, \infty)$, *then* g *is a nonnegative constant function.*

Proof. Taking $f(x) := x$, $x > 0$, and next $f(x) := -x$, $x > 0$, in (1), gives

$$g(x+y)(x+y) \leqslant g(x)x + g(y)y, \qquad x, y > 0,$$

$$g(x+y)(x+y) \geqslant g(x)x + g(y)y, \qquad x, y > 0,$$

which means that the function $G\colon (0, \infty) \to \mathbf{R}$, $G(x) := g(x)x$, is additive.

The function $f\colon (0, \infty) \to \mathbf{R}$ given by the formula

$$f(x) := \begin{cases} 0, & 0 < x \leqslant 1 \\ -x, & x > 1 \end{cases},$$

being decreasing, is subadditive in $(0, \infty)$. Substituting f into (1) we see that the function $g \cdot f$ is subadditive and $g \cdot f = 0$ in $(0,1]$. This obviously implies that $g \cdot f \leqslant 0$ in $(1, \infty)$, i.e. $G(x) = xg(x) \geqslant 0$ for all $x > 0$. Since every additive function, which is bounded below in a neighbourhood of a point is linear, we infer that there exists a constant $c \geqslant 0$ such that $G(x) = g(x)x = cx$, $x > 0$. Hence $g(x) = c \geqslant 0$ for all $x > 0$. This completes the proof.

Remark 5. Note that the analogous results can be easily proved for subadditive functions defined on (a, ∞) with $a \geqslant 0$, as well as for subadditive functions defined on $\mathbf{R}$.

A final Remark. Another type of the relation between subadditivity and monotonicity was considered by Professor Tadeusz Świątkowski and the present author in [2].

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UWAGI O MNOŻENIU FUNKCJI SUBADDYTYWNYCH

Streszczenie

Niech funkcja $g:(0, \infty) \to \mathbf{R}$ będzie nieujemna. W tej pracy dowodzi się, że jeżeli dla każdej nieujemnej funkcji subaddytywnej $f:(0, \infty) \to \mathbf{R}$, iloczyn $g \cdot f$ jest funkcją subaddytywną, to g musi być malejąca.

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