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A. Matkowska, J. Matkowski, N. Merentes

REMARK ON GLOBALLY LIPSCHITZIAN COMPOSITION OPERATORS

Introduction

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \times \mathbb{R} \to \mathbb{R}$ a fixed two-place function, and $\mathcal{F}(I)$ the linear space of all the functions $u : I \to \mathbb{R}$. The function $F : \mathcal{F}(I) \to \mathcal{F}(I)$ given by the formula

$$(F(u))(x) := f(x, u(x)), \quad x \in I, \ u \in \mathcal{F}(I),$$

is said to be a composition operator.

Let $a \in I$ be fixed. Denote by Lip(I) the Banach space of all the functions $u \in \mathcal{F}(I)$ with the norm

$$(1) \qquad \|u\|_{\mathrm{Lip(I)}} := |u(a)| + \sup\left\{\frac{u(x_1) - u(x_2)}{x_1 - x_2} : x_1, x_2 \in I;\ x_1 \neq x_2\right\}.$$

In [2] it is proved that if a composition operator F mapping Lip(I) into itself is globally Lipschitzian with respect to the Lip(I)-norm, then $f(x,y) = g(x)y + h(x)$, $(x \in I; y \in \mathbb{R})$, for some $g, h \in$Lip(I).

Next this result has been extended to some other function Banach spaces (cf. [1] for references). In particular, (cf. [3]) if F is a globally Lipschitzian selfmap of $\mathbf{C}_n(I)$, i.e. there is an $L \geq 0$ such that

$$(2) \qquad \|F(u) - F(v)\|_{C_n(I)} \leq L\|u - v\|_{C_n(I)}, \qquad u, v \in \mathbf{C}_n(I),$$

where

$$(3) \qquad \|u\|_{C_n(I)} := \sum_{i=0}^{n-1} |u^{(i)}(a)| + \sup\{|u^{(n)}(x)| : x \in I\},$$

then $f(x,y) = g(x)y + h(x)$, $(x \in I; y \in \mathbb{R})$, for some $g, h \in \mathbf{C}_n(I)$.

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In the present note we generalize these results. We show that the basic assumption of the global Lipschitz continuity of the composition operator can be essentially weakened. It turns out that the main result of [3] remains valid if the inequality (2) holds only for u, v being the polynomials at most of the degree n. Moreover, the argument presented here is much simpler than that in [3] where some complicated chain rule formulas are used.

1. Main result

We start with the following

Remark 1. Let $x_1, x_2, u_1, u_2 \in \mathbb{R}$, $x_1 \neq x_2$, and $n \in \mathbb{N}$, be arbitrarily fixed. Then it is easy to verify that the polynomial $u : \mathbb{R} \to \mathbb{R}$ given by

$$u(x) := a_n(x-a)^n + a_1 x + a_0,$$

where

$$a_n = \frac{u_1 - u_2}{k!(x_1 - x_2)}, \qquad a_1 = \frac{u_1 - u_2}{x_1 - x_2}\left(1 - \frac{(x_1-a)^n - (x_2-a)^n}{n!(x_1 - x_2)}\right),$$

$$a_0 = u_1 - \frac{u_1 - u_2}{n!(x_1 - x_2)}(x_1 - a)^n - \frac{u_1 - u_2}{x_1 - x_2}\left(1 - \frac{(x_1-a)^n - (x_2-a)^n}{n!(x_1 - x_2)}\right)x_1$$

has the following properties

$$u(x_1) = u_1, \quad u(x_2) = u_2; \qquad \|u\|_{C_n[a,b]} = |a_0| + |a_1| + \left|\frac{u_1 - u_2}{x_1 - x_2}\right|.$$

In the same way, taking arbitrary $v_1, v_2 \in \mathbb{R}$, we can find a polynomial

$$v(x) := b_n(x-a)^n + b_1 x + b_0$$

such that

$$v(x_1) = v_1, \quad v(x_2) = v_2; \quad \|v\|_{C_n[a,b]} = |b_0| + |b_1| + \left|\frac{v_1 - v_2}{x_1 - x_2}\right|.$$

where b_n, b_1, b_0 are defined as a_n, a_1, a_0 with u_1, u_2 replaced by v_1, v_2.

From the formulas for a_0, a_1, b_0, b_1 it easy to observe that

$$\|u - v\|_{C_n[a,b]} = |a_0 - b_0| + |a_1 - b_1| + \left|\frac{u_1 - u_2 - v_1 + v_2}{x_1 - x_2}\right|,$$

and, if x_1 and x_2 tend to an $x \in \mathbb{R}$, then there exists a $c(x) \in \mathbb{R}$ such that

$$\lim_{x_1, x_2 \to x} |x_1 - x_2| \|u - v\|_{C_n(I)} = c(x)|u_1 - u_2 - v_1 + v_2|. \tag{4}$$

Denote by $\mathbf{P}_n(I)$ the set of all the real polynomials of the degree at most n, restricted to the interval I.

THEOREM. *Let $F : \mathcal{F}(I) \to \mathcal{F}(I)$ be the composition operator generated by a function $f : I \times \mathbb{R} \to \mathbb{R}$, and suppose that $n, m \in \mathbb{N}$ are fixed positive integers. If F maps $\mathbf{P}_n(I)$ into $\mathbf{C}_m(I)$ and there exists an $L \geq 0$ such that*

$$(5) \qquad \|F(u) - F(v)\|_{C_m(I)} \leq L\|u - v\|_{C_n(I)}, \qquad u, v \in \mathbf{P}_n(I),$$

then there exist $g, h \in \mathbf{C}_m(I)$ such that

$$f(x, y) = g(x)y + h(x), \qquad x \in I,\ y \in \mathbb{R}.$$

Proof. We have $F(u) = f(\cdot, y)$ for each constant function $u(x) := y \in \mathbb{R}$. Since F maps $\mathbf{P}_n(I)$ into $\mathbf{C}_m(I)$, it follows that, for every $y \in \mathbb{R}$, the function $f(\cdot, y)$ is continuous in I.

From the definition of the norms (1) and (3) we get

$$\|u\|_{\mathrm{Lip(I)}} \leq \|u\|_{C_k(I)}, \qquad u \in \mathbf{C}_k(I),\ k \in \mathbb{N},$$

and inequality (5) implies

$$(6) \qquad \|F(u) - F(v)\|_{\mathrm{Lip(I)}} \leq L\|u - v\|_{C_n(I)}, \qquad u, v \in \mathbf{P}_n(I).$$

Let us fix arbitrary $x_1, x_2 \in I$, $x_1 \neq x_2$; $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and take the polynomials u and v constructed in Remark 1. Making use of the definition of Lip(I)-norm and substituting u and v to the inequality (6), we obtain

$$\left|\frac{f(x_1, u_1) - f(x_2, u_2) - f(x_1, v_1) + f(x_2, v_2)}{x_1 - x_2}\right| =$$

$$\left|\frac{f(x_1, u(x_1)) - f(x_2, u(x_2)) - f(x_1, v(x_1)) + f(x_2, v(x_2))}{x_1 - x_2}\right| \leq$$

$$\leq \|F(u) - F(v)\|_{\mathrm{Lip(I)}} \leq L\|u - v\|_{C_n(I)},$$

which implies that

$$|f(x_1, u_1) - f(x_2, u_2) - f(x_1, v_1) + f(x_2, v_2)| \leq L|x_1 - x_2|\|u - v\|_{C_n(I)}.$$

Hence, letting x_1 and x_2 tend to an arbitrary fixed $x \in I$, and making use of (4) and the continuity of $f(\cdot, y)$ for every $y \in \mathbb{R}$, we obtain

$$(7) \quad |f(x, u_1) - f(x, u_2) - f(x, v_1) + f(x, v_2)| \leq Lc(x)|u_1 - u_2 - v_1 + v_2|,$$
$$x \in I,\ u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

Substituting here $u_1 := y + w, u_2 := y, v_1 := w, v_2 := 0$, we get

$$f(x, y + w) - f(x, 0) = [f(x, y) - f(x, 0)] + [f(x, w) - f(x, 0)], \qquad y, w \in \mathbb{R}.$$

Put $h(x) := f(x, 0)$, $x \in I$. It follows that, for each fixed $x \in I$, the function $\alpha_x : \mathbb{R} \to \mathbb{R}$ given by

$$(8) \qquad \alpha_x(y) := f(x, y) - h(x), \qquad y \in \mathbb{R},$$

satisfies the Cauchy functional equation

$$\alpha_x(y + w) = \alpha_x(y) + \alpha_x(w), \qquad y, w \in \mathbb{R}.$$

Taking $v_1 = v_2 := 0$ in (7) we get

$$|\alpha_x(u_1) - \alpha_x(u_2)| \le L\, c(x)|u_1 - u_2|, \qquad u_1, u_2 \in \mathbb{R}.$$

Hence, for each $x \in I$, α_x is additive and continuous. Consequently, for each $x \in I$, there exists a $g(x) \in \mathbb{R}$ such that $\alpha_x(y) = g(x)y$, $y \in \mathbb{R}$. Now from (8) we have

$$f(x, y) - h(x) = g(x)y, \qquad x \in I,\ y \in \mathbb{R}.$$

Since $h(x) = f(x, 0) = F(0)$ and $g(x) = f(x, 1) - f(x, 0) = F(1) - F(0)$, $x \in I$, we have $g, h \in \mathbf{C}_m(I)$. This completes the proof.

Remark 2. It is easy to observe that the above Theorem remains true on replacing the norm $\|\cdot\|_{C_n(I)}$ in (6) by any norm $\|\cdot\|$ such that for some $M > 0$ and all $u \in \mathbf{P}_n(I)$, we have $\|u\| \le M\|u\|_{C_n(I)}$.

2. Some Corollaries

As an immediate consequence of Theorem we obtain

COROLLARY 1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ and let n, m be positive integer such that $m \le n$. The composition operator F generated by f maps the space $\mathbf{C}_n(I)$ into $\mathbf{C}_m(I)$ and is globally Lipschitzian, i.e. there exists an $L > 0$ such that*

$$\|F(u) - F(v)\|_{C_m(I)} \le L\|u - v\|_{C_n(I)}, \qquad u, v \in \mathbf{C}_n(I),$$

if and only if there exist $g, h \in \mathbf{C}_m(I)$ such that

$$f(x, y) = g(x)y + h(x), \qquad x \in I, y \in \mathbb{R}.$$

Remark 3. If in the above Corollary we have $m > n$, then $f(x, y) = h(x)$, $x \in I$, $y \in \mathbb{R}$ (cf. [3], also [1], p. 211, Theorem 8.3).

Because $\|u\|_{C_1(I)} = \|u\|_{\mathrm{Lip}(I)}$ for all $u \in \mathbf{P}_1(I)$, by an obvious change in the proof of Theorem 1, we obtain the following generalization of the result proved in [2] and quoted in the Introduction.

COROLLARY 2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$. If the composition operator F generated by f maps $\mathbf{P}_1(I)$ into $\mathrm{Lip}\,(I)$ and there exists an $L \geq 0$ such that*

$$\|F(u) - F(v)\|_{\mathrm{Lip}(I)} \leq L\|u - v\|_{\mathrm{Lip}(I)]}, \qquad u, v \in \mathbf{P}_1(I),$$

then there exist $g, h \in \mathrm{Lip}\,(I)$ such that

$$f(x, y) = g(x)y + h(x), \qquad x \in I,\; y \in \mathbb{R}.$$

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A. Matkowska, J. Matkowski
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
BRANCH IN BIELSKO-BIAŁA
Willowa 32
43-309 BIELSKO-BIAŁA, POLAND;

N. Merentes
ESCUELA DE FISICA Y MATEMATICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD CENTRAL DE VENEZUELA
CARACAS, VENEZUELA

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