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On (α, a)-convex functions

By

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Introduction. Let $I \subseteqq \mathbb{R}$ be an interval and $\alpha \in (0,1)$ a fixed real number. A function $f: I \to [-\infty, \infty)$ satisfying the inequality

$$f(\alpha s + (1-\alpha) y) \leqq \alpha f(s) + (1-\alpha) f(y), \quad s, y \in I,$$

is termed *α-convex*. Kuhn [10] proved that every α-convex function is Jensen convex (cf. Daróczy and Palés [5] for a simple argument). Let $\alpha, a \in (0,1)$ be fixed reals. A function $f: I \to [-\infty, \infty)$ is said to be *(α, a)-convex* iff

$$f(\alpha s + (1-\alpha) t) \leqq a f(s) + (1-a) f(t), \quad s, t \in I.$$

This notion, which for $a = \alpha$ coincides with α-convexity, was introduced by Kuhn in [11]. Some properties of (α, a)-convex functions are there established. In particular, Kuhn remarks that f must be constant if α is rational. He also mentions that he does not know any example of a nonconstant (α, a)-convex function for $\alpha \neq a$. In this context a problem of a characterization of (α, a)-convex functions in the case $\alpha \neq a$ appears in a natural way. Our main result gives a complete solution of the Kuhn problem (independently asked by S. Rolewicz, (cf. [8])).

In a recent paper Kominek [8, Theorem 2] proved the following result. If D is a convex and open subset of a linear space X endowed with a semilinear topology, and $\alpha, a \in (0,1)$ are different numbers such that at least one of them is rational, then each (α, a)-convex function $f: D \to \mathbb{R}$ is constant. In the present paper, we prove an essentially stronger fact. It turns out that the assumption that α or a is rational can be replaced by a considerably weaker condition that α and a are not algebraically conjugate.

At the end of this paper we present some corollaries for functions defined on convex and core open convex subsets of a linear space.

1. Some properties of (α, a)-affine functions and the Rodé Theorem. In this paper $\mathbb{R}$ and $\mathbb{Q}$ stand respectively for real and rational numbers.

Definition 1. Let X be a real linear space, $C \subseteqq X$ be convex, and $\alpha, a \in (0,1)$. A function $f: C \to [-\infty, \infty)$ is said to be:

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(α, a)-*convex* iff

$$f(\alpha x + (1-\alpha) y) \leqq a f(x) + (1-a) f(y), \quad x, y \in C,$$

(α, a)-*affine* iff

$$f(\alpha x + (1-\alpha) y) = a f(x) + (1-a) f(y), \quad x, y \in C.$$

Remark. Let $I \subseteq \mathbb{R}$ be an open interval, and suppose that $f: I \to [-\infty, \infty)$ is an (α, a)-convex function. It is easy to verify that if there is a $t_0 \in I$ such that $f(t_0) = -\infty$, then $f \equiv -\infty$.

The (α, a)-affine functions on $\mathbb{R}$ were extensively studied, among others, by Aczél [1], Daróczy [3], [4], Losonczi [12] (for other references cf. Aczél [2]). Actually they considered the following more general functional equation

$$A\varphi(s) + B\varphi(t) + C = \varphi(as + bt + c)$$

where $A, B, C, a, b, c \in \mathbb{R}$ are fixed. To present some results we need the following

Definition 2 (cf. for instance Kuczma [9, p. 106]). The elements $\alpha, a \in \mathbb{R}$ are said to be *conjugate* iff either they are both transcendental, or they are algebraically conjugate, i.e, they both are algebraic and have the same minimal polynomial with rational coefficients.

For $\alpha, a \in \mathbb{R}$ denote by $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(a)$ the smallest subfields of $\mathbb{R}$ containing, respectively, α and a. To prove the existence of the non-constant (α, a)-convex (and (α, a)-affine) functions we will use the following

Theorem A–D–L. *Let $\alpha, a \in \mathbb{R}$ be algebraic conjugate. Let $\mathbb{H}$ be a Hamel base of a linear space X over the field $\mathbb{Q}(\alpha)$. For every function $\varphi_0: \mathbb{H} \cup \{0\} \to \mathbb{R}$ there is an (α, a)-affine function $\varphi: X \to \mathbb{R}$ such that $\varphi|_{\mathbb{H} \cup \{0\}} = \varphi_0$.*

Proof. Since α and a are conjugate, there exists an isomorphism $\gamma: \mathbb{Q}(\alpha) \to \mathbb{Q}(a)$ such that $\gamma(\alpha) = a$ (see Kuczma [9, p. 106], Theorem 1). Take $x \in X$. Then $x = \sum_{i=1}^{n} \alpha_i h_i$, $\alpha_i \in \mathbb{Q}(\alpha)$, $h_i \in \mathbb{H}$, $i = 1, \ldots, n$, is the unique representation of x in the basis $\mathbb{H}$ of X over $\mathbb{Q}(\alpha)$ (up to terms with coefficients zero). It can easily be checked that φ given by

$$\varphi(x) := \varphi_0(0) + \sum_{i=1}^{n} \gamma(\alpha_i)\, \varphi_0(h_i),$$

is (α, a)-affine and $\varphi|_{\mathbb{H} \cup \{0\}} = \varphi_0$.

Theorem A–D–L for $X = \mathbb{R}^n$ coincides with a result of Aczél-Daróczy-Losonczi theory (cf. Kuczma [9, p. 106, Theorem 1 and p. 343, Theorem 3]).

Now, we prove the following lemma which also has an analogue among the Aczél-Daróczy-Losonczi results (cf. Kuczma [9, p. 345, Theorem 5, and p. 108, Theorem 2]).

Lemma. *Let $I \subseteq \mathbb{R}$ be an open interval and $\alpha, a \in \mathbb{R}$ be not algebraically conjugate. If $\varphi: I \to [-\infty, \infty)$ is (α, a)-affine then φ is a constant function.*

Proof. According to Remark, it is enough to show the lemma for $\varphi: I \to \mathbb{R}$. For an arbitrary fixed $t_0 \in I$ put

$$I - t_0 := \{t \in \mathbb{R}: t + t_0 \in I\},$$

and define a function $\varphi_0: (I - t_0) \to \mathbb{R}$ by the formula

$$\varphi_0(t) := \varphi(t + t_0) - \varphi(t_0), \quad t \in I - t_0.$$

It is easy to see that 0 is an interior point of $I - t_0$. By the assumption φ is (α, a)-affine. Hence we get for all $s, t \in I - t_0$

$$\begin{aligned}\varphi_0(\alpha s + (1-\alpha)t) &= \varphi(\alpha(s + t_0) + (1-\alpha)(t + t_0)) - \varphi(t_0)\\ &= a\varphi(s + t_0) + (1-a)\varphi(t + t_0) - \varphi(t_0) = a\varphi_0(s) + (1-a)\varphi_0(t),\end{aligned}$$

i.e. φ_0 is (α, a)-affine in $I - t_0$. Since $\varphi_0(0) = 0$, it follows that

$$\varphi_0(\alpha t) = a\varphi_0(t), \quad \varphi_0((1-\alpha)t) = (1-a)\varphi_0(t), \quad t \in I - t_0. \tag{1}$$

Therefore, for all $s, t \in I - t_0$ we have

$$\varphi_0(\alpha s + (1-\alpha)t) = a\varphi_0(s) + (1-a)\varphi_0(t) = \varphi_0(\alpha s) + \varphi_0((1-\alpha)t).$$

Hence, replacing αs by s and $(1-\alpha)t$ by t, we infer that φ_0 is conditionally additive in a neighbourhood C of 0, i.e.

$$\varphi_0(s + t) = \varphi_0(s) + \varphi_0(t), \quad s, t, s + t \in C.$$

Consequently $\varphi_0|_J$ can be extended to an additive function $\psi: \mathbb{R} \to \mathbb{R}$ (cf. Kuczma [9, p. 328, Theorem 3]). The function ψ is rationally homogeneous, i.e.

$$\psi(qt) = q\psi(t), \quad r \in \mathbb{Q}, t \in \mathbb{R},$$

and the first of the relations (1) implies that

$$\psi(\alpha t) = a\psi(t), \quad t \in \mathbb{R}.$$

Hence, for every polynomial P with rational coefficients, we have

$$\psi(P(\alpha)t) = P(a)\psi(t), \quad t \in \mathbb{R}. \tag{2}$$

There are two possible cases: α is either algebraic or transcendental.

In the first case, let P be the minimal polynomial of α. Since $P(a) \neq 0$ and $P(\alpha) = 0$, the equality (2) gives

$$\psi(t) = P(a)^{-1}\psi(P(\alpha)t) = P(a)^{-1}\psi(0) = 0, \quad t \in \mathbb{R}.$$

In the second case, when α is transcendental, in view of Definition 2, the number a must be algebraic. Let P be the minimal polynomial of a. Then $P(\alpha) \neq 0$, $P(a) = 0$, and by (2) we also obtain

$$\psi(t) = P(a)\psi(P(\alpha)^{-1}t) = 0, \quad t \in \mathbb{R}.$$

Thus we have shown that $\varphi_0 \equiv 0$ in a neighbourhood of 0. By definition of φ_0, the function φ is constant in a neighbourhood of t_0. Since $t_0 \in I$ was chosen arbitrarily, φ is locally constant, and, consequently, it is constant in the interval I. This completes the proof.

We need the following version of Rodé's Theorem (cf. [13]), proved by N. Kuhn in [11].

Theorem R–K. *Let $I \subseteq \mathbb{R}$ be a non-empty open interval, and $\alpha, a \in (0, 1)$ be fixed real numbers. Then for every (α, a)-convex function $f: I \to [-\infty, \infty)$ and every $t_0 \in I$, there exists an (α, a)-affine function $\varphi: I \to [-\infty, \infty)$ supporting f at the point t_0, i.e. such that*

$$f(t_0) = \varphi(t_0) \quad \text{and} \quad \varphi(t) \leqq f(t) \quad \text{for all } t \in I.$$

According to our best knowledge, N. Kuhn (cf. [10], [11]) was first to apply the powerful Rodé's Theorem in context of (α, a)-convex functions.

2. When (α, a)-convex functions are constant. The main result of this paper reads as follows:

Theorem. *Let $I \subseteq \mathbb{R}$ be an open interval and $\alpha, a \in (0, 1)$. If α, a are conjugate then there exists a nonconstant additive function $\varphi: \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi(\alpha s + (1 - \alpha) t) = a\,\varphi(s) + (1 - a)\,\varphi(t), \quad s, t \in \mathbb{R}.$$

If α, a are not conjugate then every (α, a)-convex function $f: I \to [-\infty, \infty)$, i.e. such that

$$f(\alpha s + (1 - \alpha) t) \leqq a f(s) + (1 - a) f(t), \quad s, t \in I.$$

is a constant function.

Proof. The first part is an immediate consequence of Theorem A–D–L. Suppose that α and a are not algebraically conjugate and take an $t_0 \in I$. In view of Theorem R–K there exists an (α, a)-affine function φ supporting f at the point t_0. By our Lemma the function φ is constant, so we have

$$f(t) \geqq \varphi(t) = \varphi(t_0) = f(t_0), \quad t \in I.$$

Since $t_0 \in I$ is arbitrary, it follows that f is constant in I.

Using Daróczy-Páles representation of the mean $\dfrac{s+t}{2}$ we give a simple proof of the following

Fact (Kuhn [11]). *Let C be a convex subset of a linaer space, and let $\alpha, a \in (0, 1)$ be fixed. If a function $f: I \to \mathbb{R}$ is (α, a)-convex, then it is Jensen-convex.*

Proof. From the identity (cf. Daróczy-Páles [5])

$$\frac{s+t}{2} = \alpha\left[\alpha\frac{s+t}{2} + \beta t\right] + \beta\left[\alpha s + \beta\frac{s+t}{2}\right]$$

(we write here $\beta = 1 - \alpha, b = 1 - a$) and (α, a)-convexity of the function f we have for all $s, t \in I$

$$f\left(\frac{s+t}{2}\right)=f\left(\alpha\left[\alpha\frac{s+t}{2}+\beta t\right]+\beta\left[\alpha s+\beta\frac{s+t}{2}\right]\right)$$
$$\leqq a f\left(\alpha\frac{s+t}{2}+\beta t\right)+b f\left(\alpha s+\beta\frac{s+t}{2}\right)$$
$$\leqq a^2 f\left(\frac{s+t}{2}\right)+a b f(t)+b a f(s)+b^2 f\left(\frac{s+t}{2}\right),$$

which means that

$$a(1-a) f\left(\frac{s+t}{2}\right)\leqq a(1-a)\frac{f(s)+f(t)}{2}, \quad s,t\in I,$$

and the fact follows.

Let us mention that the proof given by N. Kuhn makes use of the Rodé Theorem.

Kominek [8, Theorems 3 and 4] proved that, under weak regularity assumptions, every (α, a)-convex function, with $\alpha, a \in (0,1)$ and $\alpha \neq a$, must be constant. He applied the axiom of choice via Rodé's Theorem. We give an elementary proof of such a type of a result. Let us note that our simple method can be used to obtain more general Kominek's result.

Theorem K. *Let I be an open interval and $\alpha, a \in (0,1)$ be fixed numbers, $\alpha \neq a$. Suppose that $f: I \to \mathbb{R}$ is (α, a)-convex. Then, if f is measurable (or f is bounded above on: either a neighbourhood of a point, or on a set of positive measure, or on a set of the first category), then f is constant on I.*

Proof. By the above fact the function f is Jensen convex and Sierpiński's Theorem (resp. Bernstein-Doetsch's, Ostrowski's, or Mehdi's Theorem) (cf. Kuczma [9, p. 210]) implies that the function f is convex and continuous.

Put $\beta = 1-\alpha$ and $b = 1-a$. Let us fix an arbitrary point $t_0 \in I$ and define the function $g:(I-t_0)\to\mathbb{R}$, $g(t):=f(t_0+t)-f(t_0)$. It can be checked that g is convex and (α, a)-convex. Since $g(0)=0$, from (α, a)-convexity of g we obtain

$$g(\alpha t)\leqq a g(t), \quad g(\beta t)\leqq b g(t), \quad t\in I-t_0.$$

Hence we get

$$\text{(3)} \qquad \frac{g(\alpha t)}{\alpha t}\leqq\frac{a}{\alpha}\frac{g(t)}{t}, \quad \frac{g(\beta t)}{\beta t}\leqq\frac{b}{\beta}\frac{g(t)}{t}, \quad t\in I-t_0,\ t>0.$$

and

$$\text{(4)} \qquad \frac{g(\alpha t)}{\alpha t}\geqq\frac{a}{\alpha}\frac{g(t)}{t}, \quad \frac{g(\beta t)}{\beta t}\geqq\frac{b}{\beta}\frac{g(t)}{t}, \quad t\in I-t_0,\ t<0.$$

As a convex function, g has the one-sided derivatives at 0. So, letting t tend to 0 in the inequalities (3), we get

$$g'(0+)\leqq a\alpha^{-1}g'(0+), \quad g'(0+)\leqq b\beta^{-1}g'(0+).$$

Since $a \neq \alpha$, it follows that $g'(0+)=0$. In the same way, making use of (4), we show that $g'(0-)=0$. Thus we have proved that $g'(0)=0$. By the definition of g, we hence get $f'(t_0)=g'(0)=0$. Since t_0 has been chosen arbitrarily, this completes the proof.

3. Some corollaries for multidimensional spaces. To draw some conclusions for functions defined on subsets of general linear spaces we need a few definitions.

Let X be a linear space and D a subset of X. A point $x \in D$ is said to be *algebraically interior* to D iff for every $y \in X$ there exists an $\varepsilon > 0$ such that $x + t\,y \in D$ for all $t \in (-\varepsilon, \varepsilon)$. Put

$$\operatorname{core} D := \{x \in D\colon x \text{ is algebraically interior to } D\}.$$

A set D is said to be *algebraically open* iff $\operatorname{core} D = D$. The family of all algebraically open subsets of X is a topology in X and is called *core topology* (cf. for instance, F. A. Valentine [14], Z. Kominek, M. Kuczma [7], also E. Hille, R. S. Phillips [6, p. 14, Definition 2], where the *finitely open sets topology* is introduced).

Corollary. *Let X be a linear space and C an open in core topology convex subset of X. Suppose that $\alpha, a \in (0,1)$ are fixed.*

If α, a are conjugate then there exists a nonconstant additive function $\varphi\colon X \to \mathbb{R}$ such that

$$\varphi(\alpha x + (1-\alpha) y) = a\,\varphi(x) + (1-a)\,\varphi(y), \quad x, y \in X.$$

If α, a are not conjugate then every (α, a)-convex function $f\colon C \to [-\infty, \infty)$, i.e. such that

$$f(\alpha x + (1-\alpha) y) \leqq a\,f(x) + (1-a)\,f(y), \quad x, y \in C,$$

is a constant function.

Actually the assumption of C to be convex can be replaced by a weaker one of brokenline connectivity of C. The (α, a)-convexity of f could be understood in the following conditional sense

$$x, y, \alpha x + (1-\alpha) y \in C \Rightarrow f(\alpha x + (1-\alpha) y) \leqq a\,f(x) + (1-a)\,f(y).$$

P r o o f. If α, a are conjugate it is enough to apply Theorem A–D–L. If they are not conjugate, then take $x, y \in C$ and a broken line $x_1 \ldots x_n$ connecting $x_1 = x$ and $x_n = y$ (with the segments $x_i x_{i+1}$, $i = 1, \ldots, n-1$, contained in C). Since C is open in core topology for every segment $x_i x_{i+1}$ there is an open segment $y_i y_{i+1}$ contained in C and containing $x_i x_{i+1}$. Let us fix $i \in \{1, \ldots, n\}$, and define $g\colon (0,1) \to [-\infty, \infty)$ by

$$g(t) := f(t\,y_i + (1-t)\,y_{i+1}), \quad t \in (0,1).$$

Since f is (α, a)-convex, so is g. By our main Theorem the function g is constant. Consequently, $f(x_i) = f(x_{i+1})$. This implies that $f(x) = f(y)$ which was to be shown.

F i n a l r e m a r k. All the results formulated for linear spaces can be extended to more general case of affine spaces. The reason is that in the proofs we use only convex combinations of vectors.

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