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An Integral Jensen Inequality For Convex Multifunctions

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Abstract

We prove the following multivalued version of the Jensen integral inequality. Let X, Y be Banach spaces and $D \subset X$ an open and convex set. If $F : D \mapsto cl(Y)$ is a continuous convex function, then for each normalized measure space (Ω, Σ, μ), and for all μ-integrable functions $\phi : \Omega \mapsto D$ such that $\overline{conv\phi(\Omega)} \subset D$,

$$\int_\Omega (F \circ \phi)\, d\mu \subset F\left(\int_\Omega \phi d\mu\right).$$

Dedicated to Professor János Aczél on the occasion of his 70^{th} birthday

Introduction

Let $I \subset \boldsymbol{R}$ be an open interval and $f : I \mapsto \boldsymbol{R}$ a (continuous convex function. Then, according to the well-known integral Jensen inequality (cf. for instance Roberts-Varberg [10], p. 193, Remark J; also Kuczma [4], p. 181, Theorem 2), for each normalized measure space (Ω, Σ, μ), and for all μ-integrable functions $\phi : \Omega \mapsto I$,

$$f\left(\int_\Omega \phi d\mu\right) \leq \int_\Omega (f \circ \phi)\, d\mu$$

This inequality plays an important role in many parts of mathematics: for instance in probability theory (cf. Feller [3], p. 147), as well as in applications of some fixed point

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theorems (cf. Matkowski [5] and [7]). Let us also mention that this inequality permits to give a joint generalization of the integral versions of Hölder's and Minkowski's inequalities (cf. Matkowski [6], and Matkowski-Rätz [8]).

The main purpose of the present paper is to prove a multivalued counterpart of the above integral Jensen inequality. Let X and Y be real normed spaces, and $D \subset X$ a convex open set. Denote by $n(Y)$ the family of all nonempty subsets of Y. A set-valued function $F: D \mapsto n(Y)$ is said to be *convex* if for all $x, y \in D$, and $t \in (0,1)$,

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y).$$

(Note that F is convex if, and only if, the graph of F is a convex set in $X \times Y$). We say that a set-valued function $F: D \mapsto n(Y)$ is continuous at a point $x_0 \in D$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x) \subset F(x_0) + V \quad \text{and} \quad F(x_0) \subset F(x) + V,$$

for all $x \in (x_0 + U) \cap D$. Denote by $cl(Y)$ the family of all nonempty closed subsets of Y. In section 1 we prove that *if X, Y are Banach spaces and $F: D \mapsto cl(Y)$ is convex and continuous, then for each normalized measure space and for all μ-integrable functions $\phi: \Omega \mapsto D$ such that $\overline{conv\phi(\Omega)} \subset D$ we have*

$$\int_\Omega (F \circ \phi)\, d\mu \subset F\left(\int_\Omega \phi d\mu\right).$$

The integral of a multifunction G is understood here in the sense of R. J. Aumann, i. e. it is the set of integrals of all integrable selections of G (cf. for instance Aubin-Frankowska [1], p. 326-327).

1. An auxiliary result

For the proof of the main theorem we need the following

Lemma. *Let X be a linear topological space and suppose that $D \subset X$ is open and convex. If $F: D \mapsto cl(\boldsymbol{R})$ is a convex function, then F has one of the following forms:*

a) $F(x) = [f(x), g(x)], \quad x \in D;$

b) $F(x) = [f(x), +\infty), \quad x \in D;$

c) $F(x) = (-\infty, g(x)], \quad x \in D;$

d) $F(x) = (-\infty, +\infty), \quad x \in D,$

where $f : D \mapsto \boldsymbol{R}$ is a convex function, and $g : D \mapsto \boldsymbol{R}$ is a concave function. Moreover, if F is continuous then f and g are continuous.

Proof. First note that if $F(x_0)$ is bounded above (below) for some $x_0 \in D$, then $F(x)$ is bounded above (below) for all $x \in D$. In fact, for every $x \in D$ there exist $y \in D$ and $t \in (0,1)$ such that $tx + (1-t)y = x_0$. From the convexity of F we get the inclusion

$$tF(x) + (1-t)F(y) \subset F(x_0),$$

which clearly implies the claim.

If the values of F are bounded below, then $f : D \mapsto \boldsymbol{R}$ given by

$$f(x) := \inf F(x), \quad x \in D,$$

is well defined, and, by the convexity of F, the function f is convex. Similarly, if the values of F are bounded above, then $g : D \mapsto \boldsymbol{R}$ given by

$$g(x) := \sup F(x), \quad x \in D,$$

is well defined, and, by the convexity of F, the function g is concave.

By the convexity of F, its values are convex sets in $\boldsymbol{R}$. By assumption the values of F are also closed in $\boldsymbol{R}$. It follows that for every $x \in D$, $F(x)$ is a closed and convex interval in $\boldsymbol{R}$. Since it is easy to check that the continuity of F implies the continuity of the functions f and g, the proof is completed.

□

2. Jensen's integral inequality

The main result of this paper reads as follows:

Theorem. *Let X, Y be Banach spaces and let $D \subset X$ be open and convex. If $F : D \mapsto cl(Y)$ is a continuous convex set-valued function, then for each normalized measure space (Ω, Σ, μ), and for all μ-integrable functions $\phi : \Omega \mapsto D$ such that $\overline{\mathrm{conv}\,\phi(\Omega)} \subset D$,*

$$\int_\Omega (F \circ \phi)\, d\mu \subset F\left(\int_\Omega \phi\, d\mu\right). \tag{1}$$

Proof. We divide the proof into two steps.

Step 1. *The Theorem holds true if* $Y = \boldsymbol{R}$

In view of the Lemma, the function F is either of the form a), b), c) or d). Suppose that F has the form a), i.e. that $F(x) = [f(x), g(x)]$, $x \in D$, where $f : D \mapsto \boldsymbol{R}$ is continuous and convex and $g : D \mapsto \boldsymbol{R}$ is continuous and concave. Put

$$z := \int_\Omega \phi d\mu.$$

By the mean-value theorem (cf. Diestel-Uhl [2], p. 48, Corollary 8) and the assumption $\overline{conv\phi(\Omega)} \subset D$ we get

$$z \in \overline{conv\phi(\Omega)} \subset D.$$

Since f is continuous at z, the subdifferential $\partial f(z)$ is non-empty (cf. Phelps [9], prop. 1.11), i.e. there is a continuous linear functional $x^* : X \mapsto \boldsymbol{R}$ such that

$$f(x) \geq x^*(x - z) + f(z), \quad x \in D. \tag{2}$$

Let $h : \Omega \mapsto \boldsymbol{R}$ be an arbitrary μ-integrable selection of the composite function $F \circ \phi$. Then $h(\omega) \in F(\phi(\omega)) = [f(\phi(\omega)), g(\phi(\omega))]$, and, consequently,

$$h(\omega) \geq f(\phi(\omega)), \quad \omega \in \Omega.$$

Applying (2) we hence get

$$h(\omega) \geq x^*(\phi(\omega) - z) + f(z), \quad \omega \in \Omega.$$

Integrating both sides we obtain

$$\int_\Omega h(\omega) d\mu \geq \int_\Omega x^*(\phi(\omega) - z)\, d\mu + \int_\Omega f(z) d\mu.$$

Since

$$\int_\Omega x^*(\phi(\omega) - z)\, d\mu = x^*\left(\int_\Omega \phi(\omega) d\mu - \int_\Omega z d\mu\right) = x^*(z - z) = 0,$$

we hence get

$$\int_\Omega h(\omega) d\mu \geq \int_\Omega f(z) d\mu = f(z).$$

In the same way, taking an affine function that supports the concave function g at point z, we can show that

$$\int_\Omega h(\omega) d\mu \leq g(z).$$

Therefore

$$\int_\Omega h(\omega) d\mu \in [f(z), g(z)] = F(z) = F\left(\int_\Omega \phi d\mu\right).$$

Since the relevant arguments in cases b) and c) are even simpler, and in case d) there is nothing to prove, this completes the proof of step 1.

Step 2. *Theorem holds true for an arbitrary Banach space Y.*

Let us take an arbitrary continuous linear functional $y^* : Y \mapsto \boldsymbol{R}$ and consider the multivalued function $\overline{y^* \circ F}$ defined by the formula

$$\overline{y^* \circ F}(x) := \overline{y^*(F(x))}, \quad x \in D.$$

This function is convex and continuous, and its values are closed subsets of $\boldsymbol{R}$. From Step 1 of the proof we obtain

$$\int_\Omega \overline{y^* \circ F} \circ \phi d\mu \subset \overline{y^* \circ F}\left(\int_\Omega \phi d\mu\right). \tag{3}$$

Let $h : \Omega \mapsto Y$ be an arbitrary μ–integrable selection of $F \circ \phi$. Then $y^* \circ h$ is a μ–integrable selection of the function $y^* \circ F \circ \phi$. Consequently, in view of (3),

$$\int_\Omega y^* \circ h d\mu \in \overline{y^* \circ F}\left(\int_\Omega \phi d\mu\right).$$

Hence, making use of the relation

$$\int_\Omega y^* \circ h d\mu = y^*\left(\int_\Omega h d\mu\right),$$

we get

$$y^*\left(\int_\Omega h d\mu\right) \in \overline{y^*\left(F\left(\int_\Omega \phi d\mu\right)\right)}.$$

Since this property holds true for all continuous linear functionals y^* and the set $F\left(\int_\Omega \phi d\mu\right)$ is convex and closed, we obtain, by the separation theorem (cf. Rolewicz [11], p.98, Corollary 2.5.11):

$$\int_\Omega h d\mu \in F\left(\int_\Omega \phi d\mu\right).$$

Because h is an arbitrary integrable selection of the function $F \circ \phi$, it follows that

$$\int_\Omega F \circ \phi d\mu \subset F\left(\int_\Omega \phi d\mu\right),$$

which was to be shown.

□

Remark. Let $I \subset \boldsymbol{R}$ be an open interval and $f : I \mapsto \boldsymbol{R}$ a convex function. Taking in the above result $D := I$ and $F : D \mapsto cl(\boldsymbol{R})$ defined by

$$F(x) := [f(x), +\infty), \quad x \in D,$$

we obtain the inclusion (1) for each normalized measure space (Ω, Σ, μ), and for all μ–integrable functions $\phi : I \mapsto \boldsymbol{R}$. Since (1) implies that

$$f\left(\int_\Omega \phi d\mu\right) \le \int_\Omega (f \circ \phi) d\mu,$$

it follows that our Theorem generalizes the classical single–valued Jensen inequality.

□

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