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GEOMETRICAL CONVEXITY AND GENERALIZATIONS OF THE BOHR–MOLLERUP THEOREM ON THE GAMMA FUNCTION

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Abstract: The main result is the following. If $g : (0,\infty) \to (0,\infty)$ is geometrically convex on an interval (a,∞), for some $a \geq 0$, and satisfies the functional equation

$$g(x+1) = xg(x), \quad x \in (0,\infty); \quad g(1) = 1,$$

then g is the Γ function. This result improves the classical Bohr-Mollerup theorem. We also proved that the geometrical convexity of g on (a,∞) can be replaced by geometrical Jensen convexity on (a,∞) – i.e. $g(\sqrt{xy}) \leq \sqrt{g(x)g(y)}$ for $x, y > a$ – and some weak regularity conditions.

Introduction

The *Euler* Γ *function* is characterized as the only *logarithmically convex* function $g : (0,\infty) \to (0,\infty)$, satisfying the functional equation

$$(1) \qquad g(x+1) = x \cdot g(x), \quad x \in (0,\infty), \quad \text{with} \quad g(1) = 1\,.$$

This is the well-known theorem of H. Bohr and J. Mollerup [3], pp. 149-164, published in 1922. Nine years later E. Artin [1] gave a very elegant and easy proof of it. An elementary and nice exposition of this proof can be found in Chapter 8 of W. Rudin's book [11]. W. Krull showed in his paper [5], which he called a marginal note to Artins *"Einführung in die Theorie der Γ Funktion"*, that this result can be obtained by characterizing the convex solutions of a class of linear finite difference equations (see also M. Kuczma, [6], p. 128.)

A. E. Mayer [8] showed that in the Bohr-Mollerup theorem the condition of the logarithmical convexity cannot be replaced by that of convexity. In particular, cf. H.-H. Kairies [4], for every sufficiently small $c > 0$, the function $g : (0,\infty) \to (0,\infty)$ given by

$$g(x) = \Gamma(x)\exp(c\sin 2\pi x), \quad x \in (0,\infty),$$

satisfies (1) and is convex on $(0,\infty)$.

The convexity of a function g is meant in the classical sense. Thus g is *convex* on an interval (a,b) if and only if for each triplet of numbers $x, y, z \in (a,b)$ with $x < y$ and $x \neq z \neq y$ the following inequality holds:

$$\frac{g(x) - g(z)}{x - z} \leq \frac{g(y) - g(z)}{y - z}.$$

Moreover g *is logarithmically convex* means that $\log \circ g$ is convex. We say that a function $g : (0,\infty) \to (0,\infty)$ is *geometrically convex* if

$$g\left(x^{\lambda} \cdot y^{1-\lambda}\right) \leq g\left(x\right)^{\lambda} \cdot g\left(y\right)^{1-\lambda} \quad \text{for all} \quad \lambda \in (0,1); \quad x, y \in (0,\infty).$$

Obviously g is geometrically convex on $(0,\infty)$ if and only if its exponential conjugate, i.e. the function $\log \circ g \circ \exp$ is convex on $\mathbb{R}$.

In Section 1 we will present the following theorem:

The only function $g : (0,\infty) \to (0,\infty)$ satisfying (1) and geometrically convex on a neighbourhood of the infinity is the Γ function.

We also prove that this result essentially improves the Bohr-Mollerup theorem. In this context the above mentioned examples of convex solutions g of equation (1) such that $g \neq \Gamma$ show that the geometrical convexity is a more appropriate characterization of the Γ function than convexity or even the logarithmical convexity.

In Section 2, using some well-known weak conditions which ensure the continuity of Jensen convex functions, we give some characterizations of the Γ function under the assumption of the geometrical Jensen convexity of the function g.

1. The main result

We start this section with the following obvious remarks.

Remark 1. If the function $g : (0,\infty) \to (0,\infty)$ is a solution of (1) then $\varphi : (0,\infty) \to \mathbb{R}$, given by $\varphi = \log og$ is a solution of the functional equation

(2) $\quad \varphi(x+1) = \log x + \varphi(x), \quad x \in (0,\infty), \quad$ with $\quad \varphi(1) = 0.$

From (2) we get, by induction, for all $n \in \mathbb{N}$

(3) $\quad \varphi(n+1+x) = \varphi(x) + \log\left[x(x+1)\cdots(x+n)\right], \quad x \in (0,\infty).$

Remark 2. A function $g : (0,\infty) \to (0,\infty)$ is geometrically convex on (a,∞), $a \geq 0$, if and only if the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi = \log og \circ \exp$ is convex on $(\log a, \infty)$.

Now we can prove

Theorem 1. *Suppose that $g : (0,\infty) \to (0,\infty)$ is a solution of (1) and g is geometrically convex on an interval (a,∞) for some $a \geq 0$. Then $g \equiv \Gamma$.*

Proof. Let $g : (0,\infty) \to (0,\infty)$ be a solution of (1), geometrically convex on the interval (a,∞). Put $\varphi = \log og$ and $\phi = \log og \circ \exp$ as in Remarks 1 and 2, respectively. By Remark 2 the function ϕ is convex on $(\log a, \infty)$. Take arbitrary $n \in \mathbb{N}$ with $n > a$ and $x \in (0,1)$, and put

$$x_1 = \log n, \qquad x_2 = \log(n+1),$$
$$x_3 = \log(n+1+x), \qquad x_4 = \log(n+2).$$

So we have

$$\log a < x_1 < x_2 < x_3 < x_4.$$

From the convexity of ϕ on the interval $(\log a, \infty)$ follows:

$$\frac{\phi(x_2)-\phi(x_1)}{x_2-x_1} \leq \frac{\phi(x_3)-\phi(x_2)}{x_3-x_2} \leq \frac{\phi(x_4)-\phi(x_2)}{x_4-x_2}.$$

Since $\varphi(n) = \log\left[(n-1)!\right]$, this inequality yields

$$\frac{\log n}{x_2-x_1} \leq \frac{\varphi(n+1+x)-\log n!}{x_3-x_2} \leq \frac{\log(n+1)}{x_4-x_2}.$$

Subtracting from this inequality $\frac{\log n}{x_2-x_1}$ and multiplying by $(x_3-x_2) > 0$ yields

$$0 \leq \varphi(n+1+x) \; - \; \log n! - \frac{x_3-x_2}{x_2-x_1}\log n \leq$$
$$\leq \frac{x_3-x_2}{x_4-x_2}\log(n+1) - \frac{x_3-x_2}{x_2-x_1}\log n.$$

Put

$$\Theta_n := \frac{x_3 - x_2}{x_4 - x_2}\log(n+1) - \frac{x_3 - x_2}{x_2 - x_1}\log n.$$

Using (3) and the explicit expressions for the x_i's, we get

$$0 \le \varphi(x) - \log\left[\frac{n!}{x(x+1)\cdots(x+n)} \cdot n^{\frac{\log(n+1+x)-\log(n+1)}{\log(n+1)-\log n}}\right] \le \Theta_n.$$

We will show that $\lim_{n\to\infty} \Theta_n = 0$. The inequality $x_4 - x_2 > x_3 - x_2$ implies

$$0 \le \Theta_n < \frac{x_4 - x_2}{x_4 - x_2} \cdot \log(n+1) - \frac{x_4 - x_2}{x_2 - x_1} \cdot \log n =$$
$$= \log(n+1) - \delta(n) \cdot \log n = \log\left[\frac{n}{n^{\delta(n)}} + \frac{1}{n^{\delta(n)}}\right],$$

where $\delta(n) = \frac{x_4 - x_2}{x_2 - x_1}$. Hence we have

$$\lim_{n\to\infty} \delta(n) = \lim_{n\to\infty} \frac{\log(n+2) - \log(n+1)}{\log(n+1) - \log n} = 1.$$

Now $\delta(n) > \frac{n}{n+1}$ by Cauchy's mean value theorem and $x_4 - x_2 = = \log\frac{n+2}{n+1} < \log\frac{n+1}{n} = x_2 - x_1$, that is $1 > \delta(n)$. Hence

$$1 < \frac{n}{n^{\delta(n)}} < \frac{n}{n^{n/(n+1)}} = n^{\frac{1}{n+1}} \longrightarrow 1 \quad \text{for} \quad n \to \infty,$$

therefore

$$\lim_{n\to\infty} \log\left[\frac{n}{n^{\delta(n)}} + \frac{1}{n^{\delta(n)}}\right] = \log\left[\lim_{n\to\infty}\frac{n}{n^{\delta(n)}} + \lim_{n\to\infty}\frac{1}{n^{\delta(n)}}\right] =$$
$$= \log(1+0) = 0.$$

This means $\lim_{n\to\infty} \Theta_n = 0$.

So $\varphi(x)$ and henceforth also $g(x)$ is uniquely defined for each x of the interval $(0,1)$ and, while $\varphi(1) = 0$ by definition, also at $x = 1$. By the functional equation (1) the function g is uniquely defined on all of $(0,\infty)$. Since we know that the Γ function is geometrically convex on a neighborhood of ∞ (see Remark 4, below) the proof is complete. $\Diamond$

To show the relation between the Bohr-Mollerup theorem and our Th. 1 we need some auxiliary results.

Lemma 1. *Suppose that $g : (a,\infty) \to (0,\infty)$, $a \ge 0$, is increasing and logarithmically convex on (a,∞). Then g is geometrically convex on (a,∞).*

Proof. Take arbitrary $x, y \in (a,\infty)$. Since $a < x^\lambda y^{1-\lambda} \le \lambda x + (1-\lambda)y$ for $\lambda \in (0,1)$, making use of the monotonicity and convexity of $\log \circ g$, we have

$$\log g(x^\lambda y^{1-\lambda}) \le \log g(\lambda x + (1-\lambda)y) \le$$
$$\le \lambda \log g(x) + (1-\lambda)\log g(y) = \log\left(g(x)^\lambda g(y)^{1-\lambda}\right).$$

Hence $g(x^\lambda y^{1-\lambda}) \le g(x)^\lambda g(y)^{1-\lambda}$, i.e. the function g is geometrically convex on (a,∞). $\Diamond$

Remark 3. The function $g : (0,\infty) \to (0,\infty)$, given by

$$g(x) = \exp(-\sqrt{x+1})$$

is decreasing and logarithmically convex on $(0,\infty)$. Moreover, it is easy to verify that the function $\log \circ g \circ \exp$ is strictly concave on $\mathbb{R}$, which means that g is strictly geometrically concave on $(0,\infty)$. Thus, in Lemma 1, the supposition that g is increasing turns out to be indispensable.

Lemma 2. *If a function $g : (0,\infty) \to (0,\infty)$ satisfies* (1) *and is logarithmically convex on a neighbourhood of ∞, then there exists an $a \ge 0$ such that g is increasing on (a,∞).*

Proof. By assumption $\log \circ g$ is convex on (b,∞) for some $b \ge 0$. Thus the right derivative $(\log \circ g)'_+(x) = g'_+(x)\,(g(x))^{-1}$ exists for all $x \in (b,\infty)$ and is an increasing function on (b,∞). Suppose now that $g'_+(x) < 0$ for all $x \in (b,\infty)$. Then, of course, g would be decreasing on (b,∞). But this is a contradiction because $g(n) = (n-1)!$. Thus there exists an $a \ge b$ such that $g'_+(a)$ is nonnegative. In view of the monotonicity of $(\log \circ g)'_+ = \frac{g'_+}{g}$ we have $g'_+(x) \ge 0$ for all $x \ge a$. Consequently, the function g is increasing on (a,∞). $\Diamond$

Remark 4. The Γ function is logarithmically convex on $(0,\infty)$, see e.g. Rudin [11], p. 192. Hence, by Lemma 2 and Lemma 1, Γ is also geometrically convex at least on the interval $(2,\infty)$ (cf. the proof of Lemma 2).

The function g given in Remark 3 shows also that in Lemma 2 the assumption of g to be a solution of (1) is essential. However, the following more general result is true too: *If a function $g : (b,\infty) \to$ $\to (0,\infty)$ is logarithmically convex and is not a deacreasing function, then there exists an $a \ge b$ such that g is increasing on (a,∞).*

Now we can see, using the lemmas above, that the following result which is a generalization of the Bohr-Mollerup theorem (cf. H.-H. Kairies [4], p. 50) follows from Th. 1.

Theorem 2. *If $g : (0,\infty) \to (0,\infty)$ is a solution of* (1) *and g is logarithmically convex on an interval (a,∞) for some $a \ge 0$, then $g \equiv \Gamma$.*

Remark 5. Lemmas 1 and 2 prove that Th. 1 (as well as its consequence, Th. 2) generalizes the Bohr-Mollerup theorem. To see that Th. 1 is an essential improvement of this classical result notice that all power functions $g(x) = x^p$, $x \in (0, \infty)$, with $p > 0$, are geometrically convex but not logarithmically convex. Note also that $g(x) = x^p$ is not convex for $p \in (0, 1)$.

2. Generalizations of Theorem 1 for Jensen convex functions

We introduce the following analogue of a Jensen convex function. Let $I \subseteq (0, \infty)$ be an open interval. A function $g : I \to (0, \infty)$ is said to be *geometrically Jensen convex on I* if

$$g(\sqrt{xy}) \leq \sqrt{g(x)g(y)} \quad \text{for all} \quad x, y \in I.$$

Remark 6. It is easy to see that $g : I \to (0, \infty)$ is geometrically Jensen convex iff the function $\log \circ g \circ \exp$ is Jensen convex on the interval $\log(I)$. Furthermore, it is well-known that every continuous (geometrically) Jensen convex function is (geometrically) convex.

Using this remark and the well-known theorems of F. Bernstein-G. Doetsch [2], W. Sierpiński [12], A. Ostrowski [10] and M. R. Mehdi [9] (cf. also M. Kuczma [7]), which give some very weak sufficient conditions for a Jensen convex function to be continuous, we can formulate the theorems of the previous section in a more general form.

Corollary 1. *Suppose that $g : (0, \infty) \to (0, \infty)$ is bounded above on a neighbourhood of a point and geometrically Jensen convex on an interval (a, ∞) for some $a \geq 0$. If g satisfies (1), then $g \equiv \Gamma$.*

Proof. By assumption there are $x_0 \in (0, \infty)$, $r > 0$ and $M > 0$ such that $g(x) \leq M$ for all $x \in (x_0 - r, x_0 + r)$. Choose $n \in \mathbb{N}$ such that $(n + x_0 - r, n + x_0 + r) \subset (a, \infty)$. Hence by equation (1) we have:

$$g(x + n) = x(x + 1) \cdots (x + n - 1)g(x) \leq$$
$$\leq (x_0 + r + n)^n M, \quad x \in (x_0 - r, x_0 + r).$$

Thus g is bounded above on $U := ((x_0 + n) - r, (x_0 + n) + r) \subseteq (a, \infty)$. It follows that the function $\log \circ g \circ \exp$ is bounded above on the interval $\log(U) \subset (\log a, \infty)$. The Bernstein-Doetsch theorem (cf. Kuczma [7], p. 145, Th. 2) implies that $\log \circ g \circ \exp$ is continuous on $(\log a, \infty)$. Remark 6 yields that $\log \circ g \circ \exp$ is convex on the interval $(\log a, \infty)$, consequently, g is geometrically convex on (a, ∞). Now Cor. 1 results from Th. 1. $\Diamond$

Remark 7. It follows from Ostrowski's theorem [10] (cf. M. Kuczma [7], p. 210, Th. 1), that the Cor. 1 remains true on replacing the assumption *"g is bounded above on a neighbourhood of a point"* by *"g is bounded above on a set $T \subset (0, \infty)$ such that the inner measure of T is positive"*.

Remark 8. It follows from Mehdi's theorem [9] (cf. M. Kuczma [7], p. 210, Th. 2), that the Cor. 1 remains true on replacing the assumption *"g is bounded above on a neighbourhood of a point"* by *"there exists a set $T \subseteq (0, \infty)$ containing a second category set with the Baire property such that g is bounded above on T"*.

In a similar way as Cor. 1, using now the Sierpiński theorem [12] (cf. also M. Kuczma [7], p. 218, Th. 2), we can prove

Corollary 2. *Suppose that $g : (0, \infty) \to (0, \infty)$ is geometrically Jensen convex on an interval (a, ∞) for some $a \geq 0$, and there is a nonempty open interval $I \subset (0, \infty)$ such that the restriction $g|_I$ is measurable. If g satisfies (1), then $g \equiv \Gamma$.*

Remark 9. In M. Kuczma's book [7] one can find some other weak conditions which guarantee the continuity of Jensen convex functions. They allow to formulate somewhat more general results than the above Cors. 1 and 2.

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