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Note on a Functional Equation

The object of the present note is the discussion of the functional equation

(1) $$\Phi(z) = H[z, \Phi(f(z))],$$

where $\Phi(z)$ is the unknown function and $f(z)$ and $H(z, w)$ are complex-valued functions of complex variables.

We assume that f is analytic at $z = 0$ and

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (|z| < r),$$

(2) $$|a_1| < 1,$$

$H(z, w)$ is an analytic function of two complex variables (z, w) at the point $(0, 0)$ and $H(0, 0) = 0$.

Evidently, the necessary condition of the existence of an analytic solution of equation (1) is the existence of a formal solution.

In [1] the following theorem has been proved.

Every formal solution

(3) $$\Phi(z) = \sum_{n=1}^{\infty} c_n z^n$$

of equation (1) *has a positive radius of convergence.*

In the present note we give a proof of this theorem, shorter than that in [1].

Suppose that (3) is a formal solution of equation (1). It follows from (2) that there exists a positive integer p such that

(4) $$|H_w(0, 0)| |f'(0)|^p < 1.$$

We may write

(5) $$\Phi(z) = P(z) + z^p \varphi(z)$$

where

$$P(z) = \sum_{n=1}^{p-1} c_n z^n, \quad \varphi(z) = c_p + \sum_{n=p+1}^{\infty} c_n z^n.$$

We define the function

$$h(z,v)=\frac{H\big(z,P(f(z))+(f(z))^{p}v\big)-P(z)}{z^{p}}. \tag{6}$$

We shall prove that $h(z,v)$ is analytic at $(0,\beta)$ for arbitrary β. Suppose that

$$H(z,w)=\sum_{n=0}^{\infty}a_n(z)w^n \quad (|z|\leqslant r,\ |w|\leqslant R).$$

Let us fix $R_0>|\beta|$. Since $P(0)=0$ and $f(0)=0$, there exists $r_0\leqslant r$ such that for $|z|\leqslant r_0$ and $|v|\leqslant R_0$ we have

$$|P(f(z))+(f(z))^{p}v|\leqslant R.$$

Evidently, $f_1(z)=\dfrac{f(z)}{z}$ is analytic for $|z|\leqslant r_0$ and for $|z|\leqslant r_0$, $|v|\leqslant R_0$ we have

$$H\big(z,P(f(z))+(f(z))^{p}v\big)-P(z)=\sum_{n=0}^{\infty}a_n(z)\big[P(f(z))+z^{p}(f_1(z))^{p}v\big]^{n}-P(z)=$$

$$=\sum_{n=0}^{\infty}b_n(z)z^{np}v^{n}=b_0(z)+z^{p}\sum_{n=1}^{\infty}b_n(z)z^{p(n-1)}v^{n},$$

where $b_n(z)$ are expressed by $a_n(z)$, $f(z)$, $f_1(z)$ and $P(z)$. Since the formal series $\varphi(z)$ formally satisfies the equation

$$z^{p}\varphi(z)=b_0(z)+z^{p}\sum_{n=1}^{\infty}b_n(z)z^{p(n-1)}\big[\varphi(f(z))\big]^{n},$$

we infer that $b_0(z)=z^{p}\bar{b}(z)$ where $\bar{b}(z)$ is analytic for $|z|\leqslant r_0$. Thus $h(z,v)$ is analytic for $|z|\leqslant r_0$ and $|v|\leqslant R_0$.

By (6) we get

$$h_v'(z,v)=H_w'\big[z,P(f(z))+(f(z))^{p}v\big]\left(\frac{f(z)}{z}\right)^{p}. \tag{7}$$

Moreover, $\varphi(z)$ formally satisfies the equation

$$\varphi(z)=h\big[z,\varphi(f(z))\big] \tag{8}$$

and consequently

$$c_p=h(0,c_p). \tag{9}$$

Equation (1) together with (5) is equivalent to (8). From (7) we obtain

$$h_v'(0,c_p)=H_w'(0,0)(f'(0))^{p}.$$

Hence and from (4) there exist $\vartheta<1$, $r_1>0$ and $R_1>0$ such that for $|z|\leqslant r_1$ and $|v_1-c_p|\leqslant R_1$, $|v_2-c_p|\leqslant R_1$ we have

$$|h(z,v_2)-h(z,v_1)|\leqslant\vartheta|v_2-v_1|. \tag{10}$$

Let us fix $0 < K \leqslant R_1$. By the continuity of $h(z, v)$ there exists a number $r_4 > 0$ such that for $|z| \leqslant r_4$ we have

(11) $$|h(z, c_p) - h(0, c_p)| \leqslant (1-\vartheta)K.$$

Evidently, there exists an $r_5 > 0$ such that for $|z| \leqslant r_5$ we have

(12) $$|f(z)| \leqslant |z|.$$

Let $r = \min(r_1, r_3, r_4)$. We denote by A the set of all analytic functions $\varphi(z)$ which fulfil the condition

(13) $$|\varphi(z) - c_p| \leqslant K \quad (|z| \leqslant r, \ \varphi(0) = c_p).$$

This set with the metric $\varrho(\varphi_1, \varphi_2) = \sup\{|\varphi_2(z) - \varphi_1(z)| : |z| \leqslant r\}$ is a complete metric space.

Now we consider the transformation $\psi = T(\varphi)$ defined by the formula

(14) $$\psi(z) = h[z, \varphi(f(z))].$$

We shall prove that $T(A) \subset A$. If $\varphi \in A$, then from (14), (9), (12), (13), (10), and (11) we obtain for $|z| \leqslant r$

$$|\psi(z) - c_p| = |h[z, \varphi(f(z))] - h(0, c_p)| \leqslant |h[z, \varphi(f(z))] - h(z, c_p)| + \\ + |h(z, c_p) - h(0, c_p)| \leqslant \vartheta|\varphi(f(z)) - c_p| + (1-\vartheta)K \leqslant \vartheta K + (1-\vartheta)K = K.$$

From (9) we get $\psi(0) = c_p$, so $\psi \in A$.
Let $\varphi_1, \varphi_2 \in A$, $\psi_1 = T(\varphi_1)$, $\psi_2 = T(\varphi_2)$. From (10) we get

$$\varrho(\psi_1, \psi_2) = \sup_{|z| \leqslant r} |h[z, \varphi_2(f(z))] - h[z, \varphi_1(f(z))]| \\ \leqslant \sup_{|z| \leqslant r} |\varphi_2(f(z)) - \varphi_1(f(z))| \leqslant \vartheta\varrho(\varphi_1, \varphi_2).$$

Consequently T is a contraction. Hence, on account of the Banach contraction principle, it follows that there exists exactly one analytic solution $\varphi(z)$ of equation (8). Thus (5) is an analytic solution of equation (1).

REFERENCE

[1] W. Smajdor, *On the existence and uniqueness of analytic solutions of the functional equation* $\varphi(z) = h[z, \varphi(f(z))]$, Ann. Polon. Math. 19 (1967), 37—45.