ZESZYTY NAUKOWE POLITECHNIKI ŁÓDZKIEJ
SCIENTIFIC BULLETIN OF ŁÓDŹ TECHNICAL UNIVERSITY

Nr 687 MATEMATYKA, z. 25 1993

JANUSZ MATKOWSKI
Department of Mathematics
Technical University
Bielsko-Biała

To Professor Włodzimierz Krysicki
on his 88th birthday

# ON A GENERALIZATION OF BROWDER-GOEHDE-KIRK FIXED POINT THEOREM

Received 1992.10.26

**Abstract**

As a simple corollary of our main result we get the following generalization of Browder-Goehde-Kirk fixed point theorem for nonexpansive mappings.

Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space and T a uniformly continuous selfmapping of K. If there exists a positive sequence $t_n$, ($n\in N$), $\lim_{n\to\infty} t_n=0$ such that

$$\|x-y\|=t_n \implies \|T(x)-T(y)\|\le t_n, \quad (n\in N;\ x,y\in K),$$

then T has a fixed point in K.

## Introduction

Let X and Y be normed spaces and $K\subset X$ a convex set. Suppose that $T:K\to Y$ is a uniformly continuous mapping satisfying the following condition: there exist a $c>0$ and a sequence of positive reals $t_n$, ($n\in N$), $\lim_{n\to\infty} t_n=0$, such that

$$\|x-y\|=t_n \implies \|T(x)-T(y)\|\le ct_n, \quad (n\in N;\ x,y\in K).$$

Applying some properties of subadditive functions we prove that, under these assumptions, T is a Lipschitzian mapping with a Lipschitz constant c.

This result permits us to express many fixed point theorems for nonexpansive mappings in a more general fashion.

## 1. A property of subadditive functions

Let $\mathbb{R}_+ := (0,\infty)$. A function $f:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be subadditive if $f(s+t) \le f(s)+f(t)$, $(s,t>0)$.

The lemma below is a known fact (cf. for instance E. Hille, R. S. Phillips [6], p. 250, Theorem 7.11.1). In order to make this note self-contained, we give a short proof.

**Lemma.** _If $f:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is subadditive and $\lim_{t\to 0} f(t)=0$ then_

$$\lim_{t\to 0} \frac{f(t)}{t} = \sup\left\{ \frac{f(t)}{t} : t>0 \right\}.$$

**Proof.** Suppose that

$$\alpha := \sup\left\{ \frac{f(t)}{t} : t>0 \right\},$$

is finite. Given $\varepsilon>0$ there exists a $c>0$ such that $f(c)/c>\alpha-\varepsilon$. Let $n$ be a positive integer. For every $t$ such that $c/(n+1) \le t < c/n$ we have $cn/(n+1) \le nt < c$. Hence, by the subadditivity of $f$, $(nf(t) \ge f(nt))$,

$$\alpha \ge \frac{f(t)}{t} = \frac{nf(t)}{nt} \ge \frac{f(nt)}{nt} \ge \frac{f(c)}{nt} - \frac{f(c-nt)}{nt} \ge \frac{f(c)}{c} - \frac{f(c-nt)}{cn/(n+1)}.$$

Note that $t \longrightarrow 0$ iff $n \longrightarrow \infty$. Since $0 < c-nt \le c/(n+1)$, we have $\lim_{t\to 0}(c-nt)=0$. By the assumption, there exists $\delta>0$ such that

$$\frac{f(c-nt)}{cn/(n+1)} < \varepsilon, \qquad \text{for } t\in(0,\delta).$$

Hence, for $t\in(0,\delta)$ we have

$$\alpha \ge \frac{f(t)}{t} \ge \frac{f(c)}{c} - \varepsilon > (\alpha-\varepsilon)-\varepsilon = \alpha-2\varepsilon,$$

which completes the proof in the case $\alpha<\infty$. We omit an analogous argument in the remaining case when $\alpha=\infty$.

## 2. Remark on Lipschitzian mappings

The main result reads as follows.

**Theorem 1.** Let $X$, $Y$ be normed spaces and $K\subseteq X$ a convex subset. Suppose that $T:K\longrightarrow Y$ is uniformly continuous. If there exist $c\geq 0$ and a positive sequence $t_n$, $(n\in N)$, $\lim_{n\to\infty} t_n=0$, such that

$$\|x-y\|\leq t_n \longrightarrow \|T(x)-T(y)\|\leq ct_n, \quad (n\in N;\ x,y\in K), \tag{1}$$

then $\|T(x)-T(y)\|\leq c\|x-y\|$ for all $x,y\in K$.

**Proof.** Let $P:=\{\|x-y\|:\ x,y\in K\}$. By the convexity of $K$ we have either $P=[0,b)$ with $b\leq\infty$ or $P=[0,b]$ with $b<\infty$. Define the function $f:R_+\longrightarrow[0,\infty]$ by the formula

$$f(t):=\begin{cases} \sup\{\|T(x)-T(y)\|:\ \|x-y\|=t;\ x,y\in K\} & \text{for } t\in P \\ 0 & \text{for } t\in R_+\setminus P \end{cases}$$

The uniform continuity of $T$ and the convexity of $K$ obviously imply that $f(t)<\infty$, $(t\geq 0)$, i.e. that $f:R_+\longrightarrow R_+$. Moreover $f(0)=0$ and, by the uniform continuity of $T$, the function $f$ is continuous at 0. Take now arbitrary $s,t\in R_+$. We are going to show that

$$f(s+t)\leq f(s)+f(t), \qquad (s,t\geq 0).$$

It is clearly true if $s+t\in R_+\setminus P$. Suppose that $s+t\in P$ and take $x,y\in K$ with $\|x-y\|=s+t$. By the convexity of $K$ there exists a point $z\in K$ such that $\|x-z\|=s$ and $\|z-y\|=t$. We have

$$\|T(x)-T(y)\|\leq\|T(x)-T(z)\|+\|T(z)-T(y)\|\leq f(s)+f(t),$$

and taking the supremum over all $x,y\in K$ such that $\|x-y\|=s+t$ we obtain $f(s+t)\leq f(s)+f(t)$, which proves that $f$ is subadditive.

In view of the definition of $f$ and (1) we have $f(t_n)\leq ct_n$, $(n\in N)$, or, equivalently,

$$\frac{f(t_n)}{t_n}\leq c, \qquad (n\in N).$$

Since $\lim_{n\to\infty} t_n=0$, the Lemma implies that also

$$\sup\left\{\frac{f(t)}{t} : t>0\right\} \le c,$$

and, consequently, $f(t)\le ct$, $(t\ge 0)$. Making once more use of the definition of $f$ we hence get $\|T(x)-T(y)\|\le c\|x-y\|$ for all $x,y\in K$. This completes the proof.

**Remark 1.** A metric space $(K,d)$ is said to be *metrically convex*, (or *Menger convex*), if for every $x,y\in K$, $x\ne y$, there exists $z\in K$, $x\ne z\ne y$, such that $d(x,y)=d(x,z)+d(z,y)$. K. Menger proved that in every complete metrically convex space each two points are the endpoints of at least one metric segment (cf. L. K. Blumenthal [2], p. 41). More precisely, for every $x,y\in K$, $x\ne y$, there exists a function $F:[0,d(x,y)]\longrightarrow Y$ such that $F(0)=x$, $F(d(x,y))=y$, and for all $s,t\in[0,d(x,y)]$ we have $d(F(s),F(t))=|s-t|$.

This remark allows us to prove, similarly as Theorem 1, the following

**Theorem 2.** Let $(K,d)$ be a complete metrically convex metric space and $(Y,\rho)$ a metric space. Suppose that $T:K\longrightarrow Y$ is uniformly continuous. If there exist $c>0$ and a positive sequence $t_n$, $(n\in\mathbb{N})$, $\lim_{n\to\infty}t_n=0$, such that

$$d(x,y)=t_n \longrightarrow \rho(T(x),T(y))\le ct_n, \qquad (n\in\mathbb{N};\ x,y\in K),$$

then $\rho(T(x),T(y))\le cd(x,y)$ for all $x,y\in K$.

## 3. Some consequences for fixed point theory

The above results enable us to weaken formally the contractivity conditions in many fixed point theorems. For instance, applying Theorem 1 we get the following improvement of Browder-Goehde-Kirk theorem (cf. F. E. Browder [3], D. Goehde [5], W. A. Kirk [7], also J. Dugundji and A. Granas [4], pp. 21, 34, 35).

**Theorem 3.** Let $X$ be a uniformly convex Banach space and $K\subset X$ a nonempty closed bounded convex set. Suppose that $T:K\longrightarrow K$ is uniformly continuous. If there exists a positive sequence $t_n$, $(n\in\mathbb{N})$, $\lim_{n\to\infty}t_n=0$, such that

$$\|x-y\|=t_n \longrightarrow \|T(x)-T(y)\| \le t_n, \qquad (n\in\mathbb{N};\ x,y\in K),$$

then $T$ has a fixed point.

**Proof.** By Theorem 1 $T$ is nonexpansive and the result follows from the Browder-Goehde-Kirk fixed point theorem.

**Remark 2.** Theorem 2 allows us to improve also many fixed point theorems for set-valued mappings (cf. e.g. N. A. Assad, W. A. Kirk [1]).

## References


[1] N. A. ASSAD, W. A. KIRK (1972). *Fixed point theorems for set-valued mappings of contractive type.* Pacific J. Mat., 43, 553-562.

[2] L. E. BLUMENTHAL (1953). *Theory and applications of distance geometry.* The Clanderon Press, Oxford.

[3] F. E. BROWDER (1965). *Nonexpansive nonlinear operators in a Banach space.* Proc. NAS USA 54, 1041-1044.

[4] J. DUGUNDJI, A. GRANAS (1982). *Fixed point theory.* Vol. I, Monografie Matematyczne, Polish Scientific Publ., Warszawa.

[5] D. GOEDHE (1965). *Zum Prinzip der kontraktiven Abbildung.* Math. Nachr., 30, 251-258.

[6] E. HILLE, R. S. PHILLIPS (1957). *Functional analysis and semigroups.* Amer. Math. Soc. Colloq. Publ., 31, Amer. Math. Soc., Providence, R. I.

[7] W. A. KIRK (1965). *A fixed point theorem for mappings which do not increase distances.* Amer. Math. Monthly, 72, 1004-1006.



Department of Mathematics
Technical University
ul. Willowa 2
43-309 Bielsko-Biała
POLAND