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SOLUTIONS OF SOME FUNCTIONAL INEQUALITIES CONNECTED WITH CONVEX FUNCTIONS

JANUSZ MATKOWSKI AND KAZIMIERZ NIKODEM

Presented by J. Aczél, F.R.S.C.

Abstract. With each $T > 0$ we associate the class of functions satisfying the following functional inequality which, for $T = 1$, reduces to the definition of convex functions:

$$f(tx + (T-t)y) \le tf(x) + (T-t)f(y), \quad x, y \in J;\ t \in [0, T],$$

where $J = (0, +\infty)$ or $J = [0, +\infty)$. Making use of a sandwich type theorem we establish some relationships between these classes. It turns out that these properties depend essentially upon J.

Introduction

Let T be a fixed positive real number. Denote by J either $[0, +\infty)$ or $(0, \infty)$. In a recent paper [2] the following *sandwich type* result was proved:

Theorem 0. *A function $f : J \to \mathbf{R}$ satisfies the inequality*

$$f(tx + (T-t)y) \le tf(x) + (T-t)f(y), \quad x, y \in J;\ t \in [0, T], \tag{T}$$

if and only if there exists a convex function $g : J \to \mathbf{R}$ such that

$$T^{-1}g(Tx) \le f(x) \le g(x), \quad x \in J.$$

This theorem not only completely characterizes the solutions of the inequality (T) but gives a simple method of construction of solutions of the inequality (T) as well. For instance, taking $T \in (0, 1)$, and $g(x) = x^2$, $x \in [0, +\infty)$, we infer that every function $f : [0, +\infty) \to \mathbf{R}$ such that

$$Tx^2 \le f(x) \le x^2, \quad x \in [0, +\infty),$$

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is a solution of inequality (T). Similarly, taking $T > 1$ and $g(x) = x^{-1}$, $x \in (0,+\infty)$, we see that every function $f:(0,+\infty) \to \mathbf{R}$ such that

$$T^{-2}x^{-1} \leq f(x) \leq x^{-1}, \quad x \in (0,+\infty),$$

satisfies (T) (cf. [2]). Clearly, for $T = 1$ the set of solutions of the inequality (T) coincides with the class of convex functions.

The aim of this note is to describe how the class of all solutions of the inequality (T) depends on T. It turns out that the behaviour of these classes in the case $J = (0,+\infty)$ is essentially different from that in the case $J = [0,+\infty)$.

1. The case $J = (0,+\infty)$

Given a positive real number T, we denote by $W(T)$ the class of all solutions of (T) on the interval $J = (0,+\infty)$:

$$W(T) := \{f : (0,+\infty) \to \mathbf{R} \mid \ f \ \text{ satisfies } (T)\}.$$

The main result of this paragraph reads as follows.

Theorem 1. *If $0 < T_1 < T_2 < 1$ then $W(T_2) \subset W(T_1)$. If $1 < T_1 < T_2$ then $W(T_1) \subset W(T_2)$.*

In the proof we need the following

Lemma. *If a function $g:(0,+\infty) \to \mathbf{R}$ is convex and there exists a constant $T \in (0,1)$ (resp. $T \in (1,+\infty)$) such that*

$$g(Tx) \leq Tg(x), \quad x \in (0,+\infty), \tag{1}$$

then the function $x \to g(x)/x$ is nonde creasing (resp. nonincreasing).

Proof. Since g is convex, the function $x \to g(x)/x$ is either monotonic or there exists a point $s \in (0,+\infty)$ such that it is nonincreasing on $(0,s)$ and nondecreasing on $(s,+\infty)$ (cf. [1. Lemma]). If (1) holds with $T \in (0,1)$, then for each fixed $x_0 \in (0,+\infty)$ we have

$$\frac{g(T^n x_0)}{T^n x_0} \leq \frac{g(x_0)}{x_0}, \quad n \in \mathbf{N}, \tag{2}$$

and the sequence $(T^n z_0)$ tends to zero. Hence $g(x)/x$ cannot be decreasing on any interval $(0, s) \subset (0, +\infty)$. Therefore this function is nondecreasing on $(0, +\infty)$. In the second case, if (1) holds with $T \in (1, +\infty)$, then for every $z_0 \in (0, +\infty)$ we have also (2), but now the sequence $(T^n z_0)$ tends to $+\infty$. This implies that the function $g(x)/x$ cannot be increasing on any interval $(s, +\infty) \subset (0, +\infty)$. So $g(x)/x$ is nonincreasing on $(0, +\infty)$.

Proof of Theorem 1. Take $0 < T_1 < T_2 < 1$, and $f \in W(T_2)$. By the Theorem 0, there exists a convex function $g : (0, +\infty) \to \mathbb{R}$ such that

$$T_2^{-1} g(T_2 x) \le f(x) \le g(x), \quad x \in (0, +\infty). \tag{3}$$

Hence g satisfies (1) with $T_2 \in (0, 1)$ and, in view of the Lemma, $g(x)/x$ is nondecreasing. In particular,

$$g(T_1 x)/T_1 x \le g(T_2 x)/T_2 x, \quad x \in (0, +\infty).$$

Combining this with (3) we get

$$T_1^{-1} g(T_1 x) \le f(x) \le g(x), \quad x \in (0, +\infty),$$

which, in view of the Theorem, implies that $f \in W(T_1)$. The proof in the remaining case is analogous.

Remark 1. The class $W(1)$ (of all the convex functions on $(0, +\infty)$) is not included in any $W(T)$ for $T \ne 1$. In particular, positive constant functions do not belong to $W(T)$ with $T \in (0, 1)$, and negative constant functions do not belong to $W(T)$ with $T \in (1, +\infty)$.

2. The case $J = [0, +\infty)$

Now, given a positive real number T, put

$$W^0(T) := \{ f : [0, +\infty) \to \mathbb{R} \mid \quad f \quad \text{satisfies } (T) \quad \text{and } f(0) = 0 \}$$

We have the following

Theorem 2. *If $0 < T_1 < T_2$, then $W^0(T_2) \subset W^0(T_1)$. Moreover, for every $T > 1$,*

$$W^0(T) = \{ f : [0, +\infty) \to \mathbb{R} \mid \ f(x) = f(1)x, \ x \in [0, +\infty) \}.$$

Proof. Take $0 < T_1 < T_2$, and $f \in W^0(T_2)$. By the Theorem 0, there exists a convex function $g : [0, +\infty) \to \mathbb{R}$ such that

$$T_2^{-1} g(T_2 x) \leq f(x) \leq g(x), \quad x \in [0, +\infty). \tag{4}$$

Since $f(0) = 0$, putting here $x = 0$ we get $g(0) = 0$. It follows that the function $x \to g(x)/x$ is nondecreasing on $(0, +\infty)$ (cf. [4, Theorem 11B] or [3, Theorem 2. §3, chapter VII]). Therefore

$$g(T_1 x)/T_1 x \leq g(T_2 x)/T_2 x, \quad x \in (0, +\infty).$$

Hence, using (4), we get

$$T_1^{-1} g(T_1 x) \leq f(x) \leq g(x), \quad x \in [0, +\infty),$$

which, in view of the Theorem 0, implies that $f \in W^0(T_1)$.

Now assume that $T > 1$ and fix an $f \in W^0(T)$. By the Theorem 0, there is a convex function $g : [0, +\infty) \to \mathbb{R}$ such that

$$T^{-1} g(Tx) \leq f(x) \leq g(x), \quad x \in (0, +\infty). \tag{5}$$

Hence

$$g(Tx)/Tx \leq g(x)/x, \quad x \in (0, +\infty).$$

On the other hand, the function $x \to g(x)/x$ is nondecresing and, consequently,

$$g(x)/x \leq g(Tx)/Tx \quad x \in (0, +\infty).$$

Thus $g(Tx)/Tx = g(x)/x$ for $x \in (0, +\infty)$, and hence

$$g(T^n x)/T^n x = g(x)/x, \quad x \in (0, +\infty); \; n \in \mathbb{N}. \tag{6}$$

The monotonicity of the function $g(x)/x$ implies that the following limit exists (it may be infinite)

$$c := \lim_{x \to +\infty} (g(x)/x)$$

It follows that, for every $x \in (0, +\infty)$,

$$\lim_{n \to +\infty} g(T^n x)/T^n x = c.$$

On the other hand, in view of (6), for all $x \in (0, +\infty)$

$$\lim_{n \to +\infty} g(T^n x)/T^n x = g(x)/x.$$

Thus $g(x) = cx$, $x \in (0, +\infty)$. By (5) we infer that also $f(x) = cx$ for all $x \in [0, +\infty)$, which was to be shown.

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