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MATHEMATICS
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On the Uniqueness of Differentiable Solutions of a Functional Equation

by

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In the present paper we shall deal with the solutions of class C^r of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where φ is unknown.

The problem of the existence and uniqueness of C^r solutions of Eq. (1) was investigated by B. Choczewski [1, see also 2, chap. IV].

Let I be an interval. By $C^r[I]$ (or shortly C^r) we denote the class of functions which have a continuous derivative of order r in I.

We assume that

(i) $f(x)$ is defined and of class $C^r[I]$, $r \geq 1$, and for a certain $\xi \in I$ it fulfils the conditions

$$(f(x)-x)(\xi-x)>0 \quad \text{for} \quad x \in I, \quad x \neq \xi,$$

$$(f(x)-\xi)(\xi-x)<0 \quad \text{for} \quad x \in I, \quad x \neq \xi.$$

These conditions imply that $f(x) \in I$ for every $x \in I$, $f(\xi)=\xi$, and $\lim_{n\to\infty} f^n(x)=\xi$ for every $x \in I$ where $f^n(x)$ denotes the n-th iteration of the function $f(x)$. Hence, setting $x=\xi$ in Eq. (1) we obtain for $\eta=\varphi(\xi)$ the condition

$$\eta = h(\xi, \eta). \tag{2}$$

(ii) The function $h(x, y)$ is defined and of class C^r in a region Ω containing the point (ξ, η) such that (2) holds; for every $x \in I$ the set $\Omega_x \overset{df}{=} \{y : (x, y) \in \Omega\}$ is an open and nonempty interval; moreover, $h(f(x), \Omega_{f(x)}) \subset \Omega_x$ for every $x \in I$.

By Φ we denote the set of functions φ defined in I and such that $\varphi[f(x)] \in \Omega_x$ for $x \in I$ and $\varphi(\xi)=\eta$.

Let us define the functions $h_k(x, y, y_1, \ldots, y_k)$ by the recurrent relations

$$h_1(x, y, y_1) = \frac{\partial h}{\partial x}(x, y) + f'(x)\frac{\partial h}{\partial y}(x, y)\, y_1,$$

$$h_{k+1}(x, y, y_1, \ldots, y_{k+1}) = \frac{\partial h_k}{\partial x} + f'(x)\left(\frac{\partial h_k}{\partial y} y_1 + \ldots + \frac{\partial h_k}{\partial y_k} y_{k+1}\right).$$

We have the following lemma [see 1, also 2, p. 85]

LEMMA. *Suppose that hypotheses (i) and (ii) are fulfilled. If φ is a C^r solution of Eq. (1) in I, then we have*

$$\eta_k = h_k(\zeta, \eta, \eta_1, \ldots, \eta_k); \quad (k = 1, \ldots, r), \tag{3}$$

where

$$\eta_k = \varphi^{(k)}(\zeta); \quad (k = 1, \ldots, r). \tag{4}$$

B. Choczewski [1, see also 2, p. 90] proved that if hypotheses (i) and (ii) are fulfilled, $|f'(x)| \le 1$ in a neighbourhood of $x = \zeta$ and

$$\left|[f'(\zeta)]^r \frac{\partial h}{\partial y}(\zeta, \eta)\right| < 1, \tag{5}$$

then for any system $\eta_1, \ldots, \eta_r$ fulfilling Eqs. (3) there exists at least one C^r solution $\varphi \in \Phi$ of Eq. (1) in I fulfilling conditions (4).

We shall prove that this solution in unique. Namely, we have the following

THEOREM 1. *Suppose that hypotheses (i) and (ii) are fulfilled. If relation (5) holds, then for any system $\eta_1, \ldots, \eta_r$ fulfilling Eqs. (3) there exists at most one C^r solution $\varphi \in \Phi$ of Eq. (1) in I fulfilling conditions (4).*

Proof. Evidently, we may assume that $\zeta = \eta = 0$. Suppose that $\varphi_1, \varphi_2 \in \Phi$ are C^r solutions of Eq. (1) fulfilling conditions (4). By Taylor's formula we have

$$\varphi_i(x) = P(x) + x^r \Phi_i(x) \quad (i = 1, 2), \quad x \in I, \tag{6}$$

where $P(x) = \sum_{k=1}^{r} \frac{\eta_k}{k!} x^k$ and

$$\lim_{x \to 0} \Phi_i(x) = 0 \quad (i = 1, 2). \tag{7}$$

It is easy to verify that $\Phi_i(x)$ satisfy for $x \ne 0$, $x \in I$, the equation

$$\Phi(x) = H(x, \Phi[f(x)]), \tag{8}$$

where

$$H(x, w) = \frac{h(x, P[f(x)] + [f(x)]^r w) - P(x)}{x^r}. \tag{9}$$

The function $H(x, w)$ is evidently of class C^r for $x \neq 0$ in a vicinity of $(x, w) = (0, 0)$, and we have

$$\frac{\partial H}{\partial w}(x, w) = \left[\frac{f(x)}{x}\right]^r \frac{\partial h}{\partial y}(x, y); \quad y = P[f(x)] + [f(x)]^r w. \tag{10}$$

Hence, and by (5), it follows that there exist $a > 0$ and $b > 0$ such that for $0 < |x| \leq a$ and $|w| \leq b$ the point $(x, P[f(x)] + [f(x)]^r w)$ belongs to Ω and

$$|H(x, \overline{w}) - H(x, \overline{\overline{w}})| \leq |\overline{w} - \overline{\overline{w}}|. \tag{11}$$

Let us choose an $a_1 > 0$, $a_1 \leq a$, such that for $0 < |x| \leq a_1$ and $i = 1, 2$ we have $|\Phi_i(x)| \leq b$. By (11) we obtain for $0 < |x| \leq a_1$

$$|\Phi_1(x) - \Phi_2(x)| = \left|H\big(x, \Phi_1[f(x)]\big) - H\big(x, \Phi_2[f(x)]\big)\right| \leq |\Phi_1[f(x)] - \Phi_2[f(x)]|.$$

Hence by induction we get

$$|\Phi_1(x) - \Phi_2(x)| \leq |\Phi_1[f^n(x)] - \Phi_2[f^n(x)]| \quad \text{for} \quad n = 1, 2, \ldots$$

Since $\lim_{n\to\infty} f^n(x) = 0$, we obtain by (7) $\Phi_1(x) = \Phi_2(x)$ for $0 < |x| \leq a_1$. Hence, $\varphi_1(x) = \varphi_2(x)$ for $x \in I$ [2, p. 70, th. 3.2]. This completes the proof.

Combining Theorem 1 with the above-mentioned result of B. Choczewski, we obtain the following

Theorem 2. *Suppose that hypotheses* (i) *and* (ii) *are fulfilled. If relation* (5) *holds and* $|f'(x)| \leq 1$ *in a neighbourhood of* $x = \xi$, *then for any system* $\eta_1, \ldots, \eta_r$ *fulfilling Eqs.* (3) *there exists exactly one* C^r *solution* $\varphi \in \Phi$ *of Eq.* (1) *in* I *fulfilling conditions* (4).

B. Choczewski [1, see also 2, p. 95] obtained an analogous result without the condition on $|f'(x)|$ but assuming instead that the derivative $\partial^r h / \partial y^r$ fulfils a Lipschitz condition with respect to the variable y in a neighbourhood of (ξ, η).

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