Selected Topics in Functional Equations and Iteration Theory
Proceedings of the Austrian-Polish Seminar, Graz, 1991
Grazer Math. Ber. 316 (1992)

On some aspects of interpolation and functional equations

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Introduction. A two-place function $M : \mathbb{R}_+^2 \to \mathbb{R}_+ := (0, \infty)$, (or $M : \mathbb{R}^2 \to \mathbb{R}$), possessing the following general property:

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\}, \quad (x, y \in \mathbb{R}_+),$$

is said to be a *mean*. Using the mean function one may write down a generalization of d'Alembert's functional equation:

$$M(f(x+y), f(x-y)) = f(x)g(y) \tag{1}$$

with suitable domains, ranges and regularity conditions for f and g. With an obvious change of variables, there arises a generalization of Jensen's functional equation

$$M(f(x), f(y)) = f(\frac{x+y}{2})g(\frac{x-y}{2}).$$

A geometrical (resp. physical or numerical) interpretation of the problem formulated by (1) may be useful for some applications:

Interpolation M (of a function f) at central point x between two points $x+y$ and $x-y$ is to be expressed by exact central value $f(x)$ and weight factor $g(y)$. Depending on the choice of the interpolation method M, the functional equation (1) has certain solutions f and g. Only for these functions f and g, the chosen interpolation method is exact (not just approximate).

1991 *Mathematics Subject Classification*. Primary: 39B22, 41A05.

The three classical means: arithmetic, geometric and harmonic, are positively homogeneous, i.e.

$$M(tx, ty) = tM(x, y), \quad (t, x, y \in \mathbb{R}_+).$$

An important class of means are the so called *quasi-arithmetic means* which are of the form

$$M_h(x, y) = h^{-1}\left(\frac{h(x) + h(y)}{2}\right) \quad (x, y \in \mathbb{R}_+),$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonic bijection of $\mathbb{R}_+$, (cf. Aczél [1], p. 277, also Aczél and Dhombres [2], p. 245).

In section 1, under the assumption of the continuity of h at least at one point, we characterize the quasi-arithmetic means which are positively homogeneous. In section 2 we write down some functional equations of the form (1) with positively homogeneous quasi-arithmetic mean M.

1. Positively homogeneous quasi-arithmetic means

The main result of this section reads as follows

Theorem 1. *Let $m \in \mathbb{N}$, $m \geq 2$, and $p_1, \ldots, p_m > 0$ be fixed. Suppose that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is bijective, continuous at least at one point, and*

$$h^{-1}\Big(\sum_{k=1}^{m} p_k h(tx_k)\Big) = th^{-1}\Big(\sum_{k=1}^{m} p_k h(x_k)\Big) \tag{2}$$

for all $x_k, y_k > 0$, $(k = 1, \ldots, m)$, and $t > 0$. Then there exist $c > 0$ and $p \neq 0$, such that $h(t) = ct^p$, $(t > 0)$.

Proof. Let us fix $t > 0$ and put $h_t := h \circ (th^{-1})$. Replacing x_k by $h^{-1}(x_k)$, we can write the equation (2) in the form:

$$\sum_{k=1}^{m} p_k h_t(x_k) = h_t\Big(\sum_{k=1}^{m} p_k x_k\Big). \tag{3}$$

for all $x_k, y_k > 0$, $(k = 1, \dots, m)$.

We first show that

$$\lim_{x \to 0} h_t(x) = 0. \tag{4}$$

Suppose, for an indirect proof, that this is not true. Then there are $\varepsilon > 0$ and a sequence $u_n > 0$, such that $\lim_{n \to \infty} u_n = 0$, and

$$\varepsilon < h_t(u_n), \quad (n \in \mathbb{N}).$$

Since h_t is positive, from (3) we get the inequality

$$p_1 h_t(x) < h_t(p_1 x + y), \quad (x, y > 0).$$

Setting here $x := u_n$ we obtain

$$p_1 \varepsilon < p_1 h_t(u_n) < h_t(p_1 u_n + y), \quad (y > 0;\ n \in \mathbb{N}),$$

which obviously implies that $\varepsilon p_1 < h_t(x)$ for all $x > 0$. This is a contradiction, because h_t is a bijection of $(0, \infty)$.

Letting $x_2, \dots, x_m$ tend to zero in (3) and making use of (4) we get

$$p_1 h_t(x_1) = h_t(p_1 x_1), \quad (x_1 > 0).$$

In the same way we have

$$p_k h_t(x_k) = h_t(p_k x_k), \quad (x_1, \dots, x_m > 0).$$

Therefore by (3)

$$\sum_{k=1}^{m} h_t(p_k x_k) = h_t\Big(\sum_{k=1}^{m} p_k x_k\Big),$$

which may be written in the form

$$\sum_{k=1}^{m} h_t(x_k) = h_t\Big(\sum_{k=1}^{m} x_k\Big), \quad (x_1, \dots, x_m > 0).$$

It proves that h_t is additive. Since h_t is positive, there exists a positive real number $\phi(t)$ such that $h_t(x) = \phi(t) x$, i.e. that

$$h(t h^{-1}(x)) = \phi(t) x, \quad (t, x > 0).$$

Writing the above relation for an arbitrary $s > 0$ we have

$$h(sh^{-1}(x)) = \phi(s)x, \quad (s, x > 0).$$

Taking the compositions of functions on the left and right hand sides of the above relations we get

$$h(sth^{-1}(x)) = \phi(s)\phi(t)x, \quad (s, t, x > 0).$$

But we also have

$$h(sth^{-1}(x)) = \phi(st)x, \quad (s, t, x > 0).$$

The last two relations imply that $\phi(ts) = \phi(t)\phi(s)$, $(s, t > 0)$, i.e. ϕ is a multiplicative function. Since, by the definition of $\phi(t)$ we have

$$\phi(t) = h(th^{-1}(1)), \quad (t > 0),$$

the function ϕ is continuous at least at one point. It follows that (cf. [1], p. 41, Theorem 3, also [2], Proposition 6, p. 30), there is a $p \in \mathbb{R}$, $p \neq 0$, such that $\phi(t) = t^p$ for all $t > 0$, and consequently, $h(t) = ct^p$, $(t > 0)$, for some $c > 0$. This completes the proof.

From Theorem 1 we get

Corollary 1. *Suppose that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is bijective and continuous at least at one point. If the function $M_h : \mathbb{R}_+^2 \to \mathbb{R}_+$ given by*

$$M_h(x, y) = h^{-1}\left(\frac{h(x) + h(y)}{2}\right) \quad (x, y \in \mathbb{R}_+),$$

is positively homogeneous, i.e.

$$M_h(tx, ty) = tM_h(x, y), \quad (x, y, t > 0),$$

then the function $h(t) = ct^p$, $(t > 0)$, for some $c > 0$ and $p \in \mathbb{R}$, $p \neq 0$, and

$$M_h(x, y) = \left(\frac{x^p + y^p}{2}\right)^{1/p} \quad (x, y \in \mathbb{R}_+).$$

2. Some functional equations

Using in equation (1) a function $M := M_h$, for some bijective function $h : \mathbb{R}_+ \to \mathbb{R}_+$, we get the functional equation

$$(5) \qquad h^{-1}\left(\frac{h(f(x-y)) + h(f(x+y))}{2}\right) = f(x)g(y),$$

where $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are unknown functions. It seems to be a difficult problem to find all the solutions of this equation.

If we assume additionally that h is continuous at least at one point and M_h is positively homogeneous, then, in view of Corollary 1, the above equation takes the form

$$(f(x-y))^p + (f(x+y))^p = 2[f(x)]^p[g(y)]^p, \quad (x \in \mathbb{R};\ y \in \mathbb{R}_+).$$

Setting $F(x) := [f(x)]^p$, $x \in \mathbb{R}$, and $G(y) := [g(y)]^p$, $y \in \mathbb{R}_+$, we get

$$F(x-y) + F(x+y) = 2F(x)G(y), \quad (x \in \mathbb{R};\ y \in \mathbb{R}_+).$$

In this equation the arguments x and y are not taken from the same set. However, since $(\mathbb{R}_+, +)$ is a subset of the abelian group $(\mathbb{R}, +)$ such that $\mathbb{R} = \mathbb{R}_+ - \mathbb{R}_+$, the last functional equation can be extended to the following one (cf. [2], p. 113)

$$F(x-y) + F(x+y) = 2F(x)G(y), \quad (x, y \in \mathbb{R}).$$

This is Wilson's first generalization of d'Alembert's functional equation [4], (cf. also [1], p. 165). All the continuous solutions and the general solution are given in [1], p. 170, Theorem 1 and Theorem 2, respectively, (cf. also [3], p. 96–97).

Taking into account the possible applications mentioned in the introduction, it would be important to solve the functional equation (5) where h is not a power function.

In the context of interpolation, the stability problem of the functional equation

$$M(f(x+y), f(x-y)) = f(x)g(y),$$

where M is a mean, is especially interesting.

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