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On a-Wright convexity and the converse of Minkowski's inequality

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Summary. Let $a \in (0, \frac{1}{2}]$ be fixed. A function f satisfying the inequality

$$f(ax + (1-a)y) + f((1-a)x + ay) \leqslant f(x) + f(y),$$

called here a-Wright convexity, appears in connection with the converse of Minkowski's inequality. We prove that every lower semicontinuous a-Wright convex function is Jensen convex and we pose an open problem. Moreover, using the fact that $\frac{1}{2}$-Wright convexity coincides with Jensen convexity, we prove a converse of Minkowski's inequality without any regularity conditions.

Introduction

For a measure space (Ω, Σ, μ) denote by $S(\Omega, \Sigma, \mu)$, or for short S, the linear space of all the μ-integrable step functions $x: \Omega \to \mathbb{R}$ and by $S_+ = S_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $x \in S$. One can easily check that, for every bijective function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ := [0, \infty)$) such that $\varphi(0) = 0$, the functional $\mathbb{P}_\varphi: S \to \mathbb{R}_+$, given by the formula

$$\mathbb{P}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x| \, d\mu\right), \qquad x \in S, \tag{1}$$

is well defined. In [4] we have proved the following *converse of Minkowski's inequality*.

Suppose that (Ω, Σ, μ) *is a measure space with two sets* $B, C \in \Sigma$ *such that*

$$0 < \mu(B) < 1 < \mu(C) < \infty.$$

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If $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ *is bijective*, $\varphi(0) = 0$, φ^{-1} *is continuous at* 0 *and*

$$\mathbb{P}_\varphi(x+y) \leqslant \mathbb{P}_\varphi(x) + \mathbb{P}_\varphi(y), \qquad x, y \in S_+,$$

then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

In the proof of this result the continuity of φ^{-1} at 0 plays an important role. The problem whether this regularity assumption is indispensable is still open and seems to be difficult to decide. In this paper we attempt to replace the continuity of φ^{-1} at 0 by strengthening the assumption on the measure space. We require the existence of two disjoint sets $A, B \in \Sigma$ of positive measures $a := \mu(A)$ and $b := \mu(B)$ such that $b = 1 - a$. We show that this leads to a kind of convexity condition which, in considerably stronger form, has been already considered by E. M. Wright [7] and C. T. Ng [5]. We refer to this condition as *a-Wright convexity* and prove that every lower semicontinuous a-Wright convex function is Jensen convex. However some quite fundamental problems concerning the relationship between a-Wright convexity and Jensen convexity seem to be still open (cf. e.g. the Problem in Section 1). If $a = \frac{1}{2}$ then a-Wright and Jensen convexity coincide. This fact allows us to get rid of the continuity assumption in the converse of Minkowski's inequality in the case when $\mu(A) = \mu(B) = \frac{1}{2}$.

1. The triangle inequality for $\mathbb{P}_\varphi$ and a-Wright convexity

We begin with the following result which plays a key role in this paper.

THEOREM 1. *Let* (Ω, Σ, μ) *be a measure space with two disjoint sets* $A, B \in \Sigma$ *of positive measures* $a := \mu(A)$, $b := \mu(B)$ *such that* $b = 1 - a$. *Suppose that* $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ *is bijective and* $\varphi(0) = 0$. *If* $\mathbb{P}_\varphi$ *satisfies the triangle inequality*

$$\mathbb{P}_\varphi(x+y) \leqslant \mathbb{P}_\varphi(x) + \mathbb{P}_\varphi(y), \qquad x, y \in S_+, \tag{2}$$

then φ^{-1} *satisfies the following inequality*:

$$\varphi^{-1}(s) + \varphi^{-1}(t) \leqslant \varphi^{-1}(as + (1-a)t) + \varphi^{-1}((1-a)s + at), \qquad s, t \in \mathbb{R}_+. \tag{3}$$

Proof. For nonnegative x_1, x_2, y_1, y_2 the functions

$$x = x_1\chi_A + x_2\chi_B, \qquad y = y_1\chi_A + y_2\chi_B$$

belong to S_+ (here χ_A is the characteristic function of a set A). Putting these functions in (2) and making use of (1) we get

$$\varphi^{-1}(a\varphi(x_1+y_1)+b\varphi(x_2+y_2)) \leqslant \varphi^{-1}(a\varphi(x_1)+b\varphi(x_2)) + \varphi^{-1}(a\varphi(y_1)+b\varphi(y_2))$$

for all $x_1, x_2, y_1, y_2 \geqslant 0$. Setting $x_1 = y_2 := s$; $x_2 = y_1 := t$, $(s, t \geqslant 0)$, and taking into account that $a + b = 1$, we hence obtain the inequality

$$s + t \leqslant \varphi^{-1}(a\varphi(s)+b\varphi(t)) + \varphi^{-1}(b\varphi(s)+a\varphi(t)), \qquad (s, t \geqslant 0).$$

Now, replacing s by $\varphi^{-1}(s)$ and t by $\varphi^{-1}(t)$ we have

$$\varphi^{-1}(s)+\varphi^{-1}(t) \leqslant \varphi^{-1}(as+bt)+\varphi^{-1}(bs+at), \qquad (s, t \geqslant 0),$$

which was to be shown.

REMARK 1. A. W. Roberts and D. E. Varberg [6] referred to the inequality $f(x+r)-f(x) \leqslant f(y+r)-f(y)$ as Wright convexity of f (cf. E. M. Wright [7]). C. T. Ng [5] proved that the Wright convexity of a function $f: D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$ is convex, is equivalent to the following inequality:

$$f(\lambda x+(1-\lambda)y)+f((1-\lambda)x+\lambda y) \leqslant f(x)+f(y), \qquad \lambda \in (0,1);\ x, y \in D.$$

Taking here $\lambda = \frac{1}{2}$ we infer that every Wright convex function f is Jensen convex, i.e.,

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}, \qquad x, y \in D.$$

Let us mention that C. T. Ng proved that every Wright convex function is a sum of a convex continuous function and of an additive function (see [5]).

Motivated by Theorem 1 and by the above remark we introduce the following definition.

DEFINITION. *Let $(X, \mathbb{R}, +, \cdot)$ be a linear space, $D \subset X$ a convex set and $a \in (0, \frac{1}{2}]$ a fixed number. A function $f: D \to \mathbb{R}$ is said to be a-Wright convex (resp. a-Wright concave) if*

$$f(ax+(1-a)y)+f((1-a)x+ay) \leqslant f(x)+f(y), \qquad x, y \in D,$$

(resp. the reverse inequality holds).

REMARK 2. Theorem 1 says that φ^{-1} is a-Wright concave. Note that f is $\frac{1}{2}$-Wright convex iff f is Jensen convex, and f is Wright convex iff it is a-Wright convex for every $a \in (0, \frac{1}{2}]$.

In connection with the above definition we propose to solve the following:

PROBLEM. *Let $a \in (0, \frac{1}{2})$. Prove or disprove that every a-Wright convex function is $\frac{1}{2}$-Wright convex.*

REMARK 3. One can easily observe that every *a-convex function*, i.e. a function $f: D \to \mathbb{R}$, for which $f(ax + (1-a)y) \leqslant af(x) + (1-a)f(y)$, $(x, y \in D)$ holds $(a \in (0, 1))$, is obviously a-Wright convex. It has been proved by N. Kuhn [3] and, independently, by Z. Daróczy and Z. Páles [1] that every a-convex function is Jensen convex. In this connection one can ask if every a-Wright convex function is a-convex. The answer is no because every discontinuous additive function is a-Wright convex for every $a \in (0, \frac{1}{2})$ and cannot be a-convex for every $a \in (0, \frac{1}{2})$.

Now we are going to show that under some regularity conditions every a-Wright convex function is Jensen convex.

In the sequel we need the following obvious

REMARK 4. Let $g: [0, \frac{1}{2}] \to \mathbb{R}$ be given by the formula

$$g(u) := 2u(1-u).$$

Then: 1°. g is a homeomorphism of the interval $[0, \frac{1}{2}]$; $g(0) = 0$, $g(\frac{1}{2}) = \frac{1}{2}$; and $g(u) > u$ for every $u \in (0, \frac{1}{2})$;

2°. for every $u \in (0, \frac{1}{2})$, $\lim_{k \to \infty} g^k(u) = \frac{1}{2}$. (Here g^k is the k-th iterate of g.)

LEMMA. *Let $(X, \mathbb{R}, +, \cdot)$ be a linear space and let $D \subset X$ be a convex set. If, for an $a \in (0, \frac{1}{2})$, a function $f: D \to \mathbb{R}$ is a-Wright convex then, for every positive integer k, f is $g^k(a)$-Wright convex.*

Proof. Suppose that for an $a \in (0, \frac{1}{2})$ we have

$$f(ax + (1-a)y) + f((1-a)x + ay) \leqslant f(x) + f(y), \qquad x, y \in D.$$

Replacing in this inequality x and y by $(1-a)x+ay$ and $ax+(1-a)y$, respectively, we get

$$f(g(a)x+(1-g(a))y)+f((1-g(a))x+g(a)y) \leqslant f(x)+f(y), \qquad x, y \in D,$$

which means that f is $g(a)$-Wright convex. Now Remark 4.1° and induction completes the proof.

THEOREM 2. *Let X be a linear topological space, $D \subset X$ a convex set and $a \in (0, \frac{1}{2})$. If $f\colon D \to \mathbb{R}$ is a-Wright convex and lower semicontinuous then f is $\frac{1}{2}$-Wright convex (i.e. Jensen convex).*

Proof. From the Lemma we have

$$f(g^k(a)x+(1-g^k(a))y)+f((1-g^k(a))x+g^k(a)y) \leqslant f(x)+f(y), \qquad x, y \in D,$$

for every positive integer k. Letting $k \to \infty$ and making use of the lower semicontinuity of f and of Remark 4.2° we hence get

$$2f\left(\frac{x+y}{2}\right) \leqslant f(x)+f(y), \qquad x, y \in D,$$

which completes the proof.

2. An application

In this section we apply Theorem 1 to prove the following converse of Minkowski's inequality without any regularity assumptions.

THEOREM 3. *Let (Ω, Σ, μ) be a measure space with at least three sets $A, B, C \in \Sigma$ of positive measure and such that*

$$A \cap B = \emptyset; \qquad \mu(A) = \mu(B) = \tfrac{1}{2}; \qquad 1 < \mu(C) < \infty.$$

If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is bijective, $\varphi(0) = 0$ and

$$\mathbb{P}_\varphi(x+y) \leqslant \mathbb{P}_\varphi(x) + \mathbb{P}_\varphi(y), \qquad x, y \in S_+,$$

then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.

Proof. According to Theorem 1 the function φ^{-1} is $\frac{1}{2}$-Wright concave. Consequently (cf. Remark 2) it is Jensen concave. Since φ^{-1}, being nonnegative, is bounded from below, it follows from the Bernstein–Doetsch Theorem (cf. M. Kuczma [2], p. 145) that it is a concave homeomorphism of $(0, \infty)$. Clearly φ^{-1} must be increasing and, therefore, continuous at 0. Now our theorem follows from the converse of Minkowski's inequality quoted in the introduction.

REMARK 5. If $\mu(\Omega) < \infty$ then, for every bijective $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ and for every $x \in S$, we have $\varphi \circ |x| \in S_+$. Therefore the condition $\varphi(0) = 0$ in the definition of the functional $\mathbb{P}_\varphi$ is superfluous. Let us note that in this case the assumption $\varphi(0) = 0$ in Theorem 3, as well as in the converse of Minkowski's inequality, is a consequence of the triangle inequality. To show this, put $c := \mu(\Omega)$. Taking in the triangle inequality $x = y = t\chi_\Omega$ we get

$$\varphi^{-1}(c\varphi(2t)) \leqslant 2\varphi^{-1}(c\varphi(t)), \qquad (t \geqslant 0),$$

i.e., $f(2t) \leqslant 2f(t)$, $(t \geqslant 0)$, where $f := \varphi^{-1} \circ (c\varphi)$. Since $c > 0$, the function f is bijective. Consequently there exists a $t_0 \geqslant 0$ such that $f(t_0) = 0$. The inequality $f(2t) \leqslant 2f(t)$ implies that also $f(2t_0) = 0$. Since f is one-to-one it follows that $t_0 = 0$. Thus $f(0) = \varphi^{-1}(c\varphi(0)) = 0$, and, since $c \neq 1$, $\varphi(0) = 0$.

Modifying the definition of the functional $\mathbb{P}_\varphi$ to

$$\mathbb{P}_\varphi(x) := \varphi^{-1}\left(\int_{\Omega_x} \varphi \circ |x| \, d\mu\right), \qquad x \in S,$$

where $\Omega_x := \{\omega \in \Omega: x(\omega) \neq 0\}$, and making use of this remark, one can drop the assumption $\varphi(0) = 0$ even in the case $\mu(\Omega) = \infty$.

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