Math. Japonica 33, No. 1 (1988), 81–86.

ON NEMYTSKII OPERATOR

Janusz Matkowski*

(Received May 12, 1986 ; revised December 15, 1986)

Abstract. Let $(X, |\cdot|)$, $(Y, |\cdot|)$, $(Z, |\cdot|)$ be normed spaces and let $U \subset X$ be a convex set. Denote by $\mathrm{Lip}(U, Y)$ the normed space of Lipschitz functions mapping U into Y. It is shown that every globally Lipschitzian Nemytskii operator mapping $\mathrm{Lip}(U, Y)$ into $\mathrm{Lip}(U, Z)$ is generated by a function $h : U \times Y \to Z$ of the form $h(x, y) = A(x)y + b(x)$, $(x \in U, y \in Y)$, where $A : U \to L(Y, Z)$ and $b \in \mathrm{Lip}(U, Z)$.

Let $(X, |\cdot|)$, $(Y, |\cdot|)$, $(Z, |\cdot|)$ be normed spaces and let $U \subset X$. We may assume, without any loss of generality, that $O \in U$. Denote by $F(U, Y)$ the vector space of all functions $\varphi : U \to Y$ and by $\mathrm{Lip}(U, Y)$ the vector space of all functions $\varphi \in F(U, Y)$ such that

$$\sup_{x \neq \bar{x}} \frac{|\varphi(x) - \varphi(\bar{x})|}{|x - \bar{x}|} < \infty ,$$

where supremum is taken over all $x, \bar{x} \in U$. Clearly, $\mathrm{Lip}(U, Y)$ with the norm defined by the formula

$$(1) \qquad \|\varphi\| := |\varphi(O)| + \sup_{x \neq \bar{x}} \frac{|\varphi(x) - \varphi(\bar{x})|}{|x - \bar{x}|}$$

is a normed space. In the same way we introduce the normed space $\mathrm{Lip}(U, Z)$.

Denote by $(L(Y, Z), \|\cdot\|)$ the normed space of all linear and continuous mappings $A : Y \to Z$. Consequently, $\|A\|$ denotes the operator norm of A.

Every function $h : U \times Y \to Z$ generates the so called Nemytskii operator or operator of substitution $N = N_h : F(U, Y) \to F(U, Z)$ defined by the formula

$$(2) \qquad (N\varphi)(x) := h(x, \varphi(x)), \quad (x \in U, \varphi \in F(U, Y)).$$

AMS (1980) Subject Classification. Primary 47H99, 46E30.
Key words and phrases. Nemytskii operator, Lipschitz operator, normed space, Banach space, linear mapping, Banach-Steinhaus theorem.

* Department of Mathematics, Technical University, 43–300 Bielsko-Biała, Poland.

This operator occurs frequently in connection with integral equations and iterative functional equations.

The main result of this paper reads as follows.

Theorem 1. *Let* $(X, |\cdot|)$, $(Y, |\cdot|)$, $(Z, |\cdot|)$ *be normed spaces and suppose that* $U \subset X$ *is convex. If the Nemytskii operator* N *defined by* (2) *maps* $\mathrm{Lip}(U, Y)$ *into* $\mathrm{Lip}(U, Z)$ *and is a globally Lipschitzian map, i.e. there is a* $c \geq 0$ *such that*

$$(3) \qquad \|N(\varphi_1) - N(\varphi_2)\| \leq c\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \mathrm{Lip}(U, Y),$$

then there exist functions $A : U \to L(Y, Z)$ *and* $b \in \mathrm{Lip}(U, Z)$ *such that*

$$(4) \qquad h(x, y) = A(x)y + b(x), \quad x \in U, \ y \in Y.$$

If, moreover, $(Y, |\cdot|)$ *is a Banach space then* $A \in \mathrm{Lip}(U, L(Y, Z))$.

Proof. Since for every fixed $y \in Y$ the function $\varphi(x) := y$, $x \in U$; being constant, belongs to $\mathrm{Lip}(U, Y)$, it follows that

$$h(\cdot, y) \in \mathrm{Lip}(U, Z), \quad y \in Y.$$

Consequently, h is continuous with respect to the first variable for every fixed $y \in Y$.

Using definition (1) we may write assumption (3) in the following form

$$|h(0, \varphi_1(0)) - h(0, \varphi_2(0))| + \sup_{t \neq \bar{t}} \frac{|h(t, \varphi_1(t)) - h(t, \varphi_2(t)) - h(\bar{t}, \varphi_1(\bar{t})) + h(\bar{t}, \varphi_2(\bar{t}))|}{|t - \bar{t}|} \leq c\|\varphi_1 - \varphi_2\|$$

where supremum is taken over all $t, \bar{t} \in U$. Hence it follows that

$$(5) \qquad \frac{|h(t, \varphi_1(t)) - h(t, \varphi_2(t)) - h(\bar{t}, \varphi_1(\bar{t})) + h(\bar{t}, \varphi_2(\bar{t}))|}{|t - \bar{t}|} \leq c\|\varphi_1 - \varphi_2\|$$

for all $\varphi_1, \varphi_2 \in \mathrm{Lip}(U, Y)$ and $t, \bar{t} \in U$; $t \neq \bar{t}$.

Let us fix $x, \bar{x} \in U$, $x \neq 0$, such that $|\bar{x}| < |x|$ and $y_1, y_2, \bar{y}_1, \bar{y}_2 \in Y$ and define the function

$$
(6) \qquad \varphi_i(t) := \begin{cases} \bar{y}_i & |t| \leq |\bar{x}| \\ \dfrac{y_i - \bar{y}_i}{|x| - |\bar{x}|}(|t| - |\bar{x}|) + \bar{y}_i, & |\bar{x}| \leq |t| \leq |x| \\ y_i & |t| > |x| \end{cases}
$$

for $t \in U$ and $i = 1, 2$. Evidently $\varphi_1, \varphi_2 \in \mathrm{Lip}(U, Y)$ and

$$
\| \varphi_1 - \varphi_2 \| = |\bar{y}_1 - \bar{y}_2| + \frac{|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|}{|x| - |\bar{x}|}.
$$

Hence setting in (5) φ_1, φ_2 defined by (6) and $t := x$, $\bar{t} := \bar{x}$ we obtain the inequality

$$
\frac{|h(x, y_1) - h(x, y_2) - h(\bar{x}, \bar{y}_1) + h(\bar{x}, \bar{y}_2)|}{|x - \bar{x}|} \leq c\left(|\bar{y}_1 - \bar{y}_2| + \frac{|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|}{|x| - |\bar{x}|}\right),
$$

which can be rewritten in the following form

$$
(7) \qquad \begin{aligned} &|h(x, y_1) - h(x, y_2) - h(\bar{x}, \bar{y}_1) + h(\bar{x}, \bar{y}_2)| \leq \\ &c\left(|\bar{y}_1 - \bar{y}_2|\,|x - \bar{x}| + \frac{|x - \bar{x}|}{|x| - |\bar{x}|}\,|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|\right). \end{aligned}
$$

Since $\dfrac{|x - \bar{x}|}{|x| - |\bar{x}|} \geq 1$ and $O \in U$ it follows from the convexity of U that

$$
\lim_{\substack{\bar{x} \to x \\ \bar{x}, x \in U}} \inf \frac{|x - \bar{x}|}{|x| - |\bar{x}|} = 1.
$$

(In fact, it is enough to choose $\bar{x}=tx,\ 0<t<1$). Hence, taking the lim inf as $\bar{x}\to x$ in the both sides of (7), we get on using of the continuity of $h(\cdot,y)$

(8) $|h(x,y_1)-h(x,y_2)-h(x,\bar{y}_1)+h(x,\bar{y}_2)|\leqslant c|y_1-y_2-\bar{y}_1+\bar{y}_2|,$

$$x\neq 0\ x\in U,\ y_1,y_2,\bar{y}_1,\bar{y}_2\in Y.$$

By the continuity of $h(\cdot,y)$ it follows that (8) also holds for $x=0$. Let us fix an $x\in U$ and define the function $A(x): Y\to Z$ by the formula

(9) $$A(x)(y):=h(x,y)-h(x,0),\quad y\in Y.$$

Taking in (8) $y_1:=y+w,\ y_2:=y,\ \bar{y}_1:=w,\ \bar{y}_2:=0$ we get

$$h(x,y+w)-h(x,y)-h(x,w)+h(x,0)=0,\quad y,w\in Y,$$

which means that

$$A(x)(y+w)=A(x)(y)+A(x)(w),\quad y,w\in Y,$$

i.e. $A(x)$ is an additive mapping. Setting $\bar{y}_1=\bar{y}_2=0$ in (8) we obtain

$$|A(x)(y_1)-A(x)(y_2)|\leqslant c|y_1-y_2|,\quad y_1,y_2\in Y,$$

which implies the continuity of $A(x)$. Therefore $A(x)\in L(Y,Z)$ and $\|A(x)\|\leqslant c$ for all $x\in U$. Putting

$$b(x):=h(x,0),\quad x\in U,$$

we have, according to (9),

$$h(x,y)=A(x)y+b(x),\quad x\in U,\ y\in Y,$$

with $A\in F(U,L(Y,Z))$ and $b\in \mathrm{Lip}(U,Z)$.

Suppose now that $(Y,|\cdot|)$ is a Banach space. We have to prove that $A\in \mathrm{Lip}(U,L(Y,Z))$, i.e. that

(10) $$M:=\sup_{\substack{x\neq\bar{x}\\ x,\bar{x}\in U}}\frac{\|A(x)-A(\bar{x})\|}{|x-\bar{x}|}<\infty.$$

To this end consider the following family of linear and continuous mappings

$$(11) \qquad \left(\frac{A(x)-A(\bar{x})}{|x-\bar{x}|}\right) \quad x, \bar{x} \in U;\ x \neq \bar{x}.$$

Clearly, for every $x, \bar{x} \in U$, $x \neq \bar{x}$, we have $\frac{A(x)-A(\bar{x})}{|x-\bar{x}|} \in L(Y, Z)$.

Since $N : \mathrm{Lip}(U, Y) \to \mathrm{Lip}(U, Z)$ and for every fixed $y \in Y$, the constant function $\varphi(x) := y$, $x \in U$, belongs to $\mathrm{Lip}(U, Y)$, it follows that $N(\varphi) - b$ belongs to $\mathrm{Lip}(U, Z)$. From the just proved part of the theorem we have $N(\varphi) - b = A(\cdot)y$. Consequently, for every $y \in Y$, $A(\cdot)y \in \mathrm{Lip}(U, Z)$, and, therefore

$$\sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \frac{|A(x)y - A(\bar{x})y|}{|x-\bar{x}|} = \sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \left| \frac{A(x)-A(\bar{x})}{|x-\bar{x}|} y \right| < \infty, \ y \in Y.$$

This means that family (11) is pointwise bounded. According to the Banach-Steinhaus theorem the number M in (10) is finite. This completes the proof.

Remark 1. One can easily verify that instead of convexity of U it is sufficient to assume that U is starshaped with respect to the point $O \in U$.

Theorem 1 gives some necessary conditions for Nemytskii operator to be a globally Lipschitzian map from $\mathrm{Lip}(U, Y)$ into $\mathrm{Lip}(U, Z)$. The next theorem provides some sufficient conditions.

Theorem 2. *Let $(X, |\cdot|)$, $(Y, |\cdot|)$, $(Z, |\cdot|)$ be normed spaces and let U be a bounded subset of X such that $O \in U$. Suppose that $A \in \mathrm{Lip}(U, L(Y, Z))$, $b \in \mathrm{Lip}(U, Z)$ and $h : U \times Y \to Z$ is defined by the formula $h(x, y) := A(x)y + b(x)$, $x \in U$, $y \in Y$.*
Then the Nemytskii operator N generated by the function h maps $\mathrm{Lip}(U, Y)$ into $\mathrm{Lip}(U, Z)$ and is globally Lipschitzian. Moreover,

$$M := \sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \frac{\| A(x) - A(\bar{x}) \|}{|x-\bar{x}|} < \infty \ ; \quad a := \sup_{x \in U} \| A(x) \| < \infty$$

and

$$\| N(\varphi_1) - N(\varphi_2) \| \leqslant (\alpha + \beta M) \| \varphi_1 - \varphi_2 \|, \qquad \varphi_1, \varphi_2 \in \mathrm{Lip}(U, Y),$$

where $\beta := \max(1, \operatorname{diam} U)$.

The proof is very simple so we omit it.

Remark 2. Note that the boundeness condition of the set U in Theorem 2 cannot be omitted. To see this take $X = Y = Z = R$ with the usual norm, $U = [0, \infty)$ and $h(x, y) = (\sin x) y$, $(x \in U, y \in R)$. We have here $A(x) = \sin x$, $b = 0$, and, evidently, $A \in \mathrm{Lip}(U, L(R, R))$. The function $\varphi(x) = x$, $x \in U$, belongs to $\mathrm{Lip}(U, R)$ and $N(\varphi)$ does not belong to this space. Clearly, N cannot be Lipschitzian. On the other hand, if we replace the norm (1) by the following stronger one

$$\| \varphi \| := \sup_{x \in U} | \varphi(x) | + \sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \frac{| \varphi(x) - \varphi(\bar{x}) |}{| x - \bar{x} |}$$

then the assumption of the boundeness of the set U in Theorem 2 is redundant.

Remark 3. The special case of Theorem 1 when $X = Y = Z = R$, $U = [a, b]$, has been proved in [1].

Reference

[1] Matkowski, J.: Functional equations and Nemytskii operators, Funkcialaj Ekvacioj 25 (1982), 127–132.