On Nemytskiĭ Lipschitzian Operator

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In an earlier author's paper it has been proved that every Nemytskiĭ operator N mapping the Banach space of Lipschitzian functions into itself and globally Lipschitzian with respect to the Lip-norm has to be of the form $N(\varphi)(x) = A(x)\varphi(x) + B(x)$ where A and B are given Lipschitzian functions. In this paper we give a kind of local version of this result.

1. It has been proved in [3] that every Nemytskiĭ operator N mapping $\mathrm{Lip}[a, b]$ into itself and globally Lipschitzian with respect to the $\mathrm{Lip}[a, b]$-norm has to be of the form

$$N(\varphi)(x) = A(x)\varphi(x) + B(x), \quad x \in [a, b], \quad \mathrm{Lip}[a, b],$$

where $A, B \in \mathrm{Lip}[a, b]$. Recently this result has been extended to the Nemytskiĭ operators mapping a normed space $\mathrm{Lip}(U, Y)$ into $\mathrm{Lip}(U, Z)$ where Y and Z are normed spaces and U is a convex (or starshaped) subset of a normed space X (cf. [4]).

Similar theorems have also been proved for the Banach spaces $BV[a, b]$, $C^r[a, b]$ and $\mathrm{Lip}^r[a, b]$ (cf. [5], [6], [7]).

In the present paper we give a kind of local version of the above result. This "locality" is understood here in the sense of the supremum norm, i.e. a weaker one than any of the norms of Banach spaces mentioned above.

2. Let $(X, |\cdot|)$, $(Y, |\cdot|)$, $(Z, |\cdot|)$ be normed spaces and let $U \subset X$. Denote by $F(U, Y)$ the vector space of all functions $\varphi: U \to Y$ and by $\mathrm{Lip}(U, Y)$ the vector space of all functions $\varphi \in (U, Y)$ such that

$$\sup_{x \neq \bar{x}} \frac{|\varphi(x) - \varphi(\bar{x})|}{|x - \bar{x}|} < \infty,$$

where supremum is taken over all $x, \bar{x} \in U$. Assume that $0 \in U$. Clearly, $\mathrm{Lip}(U, Y)$ with the norm defined by the formula

$$\|\varphi\| := |\varphi(0)| + \sup_{x \neq \bar{x}} \frac{|\varphi(x) - \varphi(\bar{x})|}{|x - \bar{x}|} \tag{1}$$

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is a normed space. Let

$$\|\varphi\|_\infty := \sup_{x \in U} |\varphi(x)|, \quad \varphi \in \mathrm{Lip}(U, Y)$$

and let $(L(Y, Z), \|\cdot\|)$ be the normed space of all linear and continuous mappings $A\colon Y \to Z$.

Every function $h\colon U \times Y \to Z$ generates the so called Nemytskii operator $N = N_h\colon F(U, Y) \to F(U, Z)$ defined by the formula

$$N(\varphi)(x) := h(x, \varphi(x)), \quad x \in U, \quad \varphi \in F(U, Y). \tag{2}$$

In general it is, of course, a nonlinear operator.

We are going to prove the following

Theorem. *Let $(X, |\cdot|)$, $(Y, |\cdot|)$, $(Z, |\cdot|)$ be normed spaces and suppose that $U \subset X$ is star-shaped with respect to 0. If the Nemytskii operator N defined by (2) satisfies for a positive number r the following two conditions:*

1°. $N\colon \{\varphi \in \mathrm{Lip}(U, Y)\colon \|\varphi\|_\infty \leq r\} \to \mathrm{Lip}(U, Z)$;

2°. *there is a $c \geq 0$ such that*

$$\|N(\varphi_1) - N(\varphi_2)\| \leq c\|\varphi_1 - \varphi_2\|, \quad \varphi_i \in \mathrm{Lip}(U, Y), \quad \|\varphi_i\|_\infty \leq r, \tag{3}$$

then there exist functions $A\colon U \to L(Y, Z)$ and $B \in \mathrm{Lip}(U, Y)$ such that

$$h(x, y) = A(x)\, y + B(x), \quad x \in U, \quad y \in Y, \quad |y| \leq r. \tag{4}$$

If, moreover, $(Y, |\cdot|)$ is a Banach space then $A \in \mathrm{Lip}(U, L(Y, Z))$.

Proof. Since for every fixed $y \in Y$ the constant function $\varphi(x) = y$, $x \in U$, belongs to $\mathrm{Lip}(U, Y)$, it follows from 1° that

$$h(\cdot, y) \in \mathrm{Lip}(U, Y), \quad y \in Y, \quad |y| \leq r.$$

Therefore h is continuous with respect to the first variable for every fixed y from the ball $B(0, r) := \{y \in Y\colon |y| \leq r\}$.

Using definition (1) we may write assumption (3) in the following form

$$|h(0, \varphi_1(0)) - h(0, \varphi_2(0))| + \sup_{t \neq \bar t} \frac{|h(t, \varphi_1(t)) - h(t, \varphi_2(t)) - h(\bar t, \varphi_1(\bar t)) + h(\bar t, \varphi_2(\bar t))|}{|t - \bar t|} \leq c\|\varphi_1 - \varphi_2\|$$

where supremum is taken over all $t, \bar t \in U$ and $\|\varphi_i\|_\infty \leq r$, $i = 1, 2$. Hence it follows that

$$\frac{|h(t, \varphi_1(t)) - h(t, \varphi_2(t)) - h(\bar t, \varphi_1(\bar t)) + h(\bar t, \varphi_2(\bar t))|}{|t - \bar t|} \leq c\|\varphi_1 - \varphi_2\| \tag{5}$$

for all $\varphi_1, \varphi_2 \in \mathrm{Lip}(U, Y)$ such that $\|\varphi_i\|_\infty \leq r$, $i = 1, 2$ and $t, \bar t \in U$, $t \neq \bar t$.

Let us fix $x \in U$, $x \neq 0$, and $\bar{x}$ from the segment joining 0 with x. Take $y_1, y_2, \bar{y}_1, \bar{y}_2 \in B(0, r)$ and define the functions

$$\varphi_i(t) := \begin{cases} \bar{y}_i & |t| < |\bar{x}| \\ \dfrac{y_i - \bar{y}_i}{|x| - |\bar{x}|}(|t| - |x|) + y_i, & |\bar{x}| \leqq |t| \leqq |x| \\ y_i & |t| > |x| \end{cases} \tag{6}$$

for $t \in U$ and $i = 1, 2$. Evidently $\varphi_i \in \operatorname{Lip}(U, Y)$, $\|\varphi_i\|_\infty \leqq r$, $i = 1, 2$, and

$$\|\varphi_1 - \varphi_2\| = |y_1 - y_2| + \frac{|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|}{|x| - |\bar{x}|}.$$

Hence, setting in (5) φ_1, φ_2 defined by (6) and $t := x$, $\bar{t} := \bar{x}$, we obtain the inequality

$$\frac{|h(x, y_1) - h(x, y_2) - h(\bar{x}, \bar{y}_1) + h(\bar{x}, \bar{y}_2)|}{|x - \bar{x}|} \leqq c\left(|y_1 - y_2| + \frac{|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|}{|x| - |\bar{x}|}\right),$$

which can be rewritten in the following form

$$|h(x, y_1) - h(x, y_2) - h(\bar{x}, \bar{y}_1) + h(\bar{x}, \bar{y}_2)| \leqq$$
$$\leqq c\left(|y_1 - y_2|\,|x - \bar{x}| + \frac{|x - \bar{x}|}{|x| - |\bar{x}|}\,|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|\right).$$

Letting $\bar{x}$ tend to x, using of the continuity of $h(\cdot, y)$, we hence get

$$|h(x, y_1) - h(x, y_2) - h(x, \bar{y}_1) + h(x, \bar{y}_2)| \leqq c|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|, \tag{7}$$

for $x \neq 0$, $x \in U$, $y_1, y_2, \bar{y}_1, \bar{y}_2 \in B(0, r)$.

By the continuity of $h(\cdot, y)$ it follows that (7) holds for $x = 0$. Let us fix an $x \in U$ and define the function $A(x)\colon B(0, r) \to Z$ by the formula

$$A(x)(y) := h(x, y) - h(x, 0). \tag{8}$$

Taking in (7) $y_1 := y + w$, $y_2 := y$, $\bar{y}_1 := w$, $\bar{y}_2 := 0$ such that $y, w \in B(0, r/2) \subset Y$ we obtain

$$h(x, y + w) - h(x, y) - h(x, w) + h(x, 0) = 0,$$

which means that

$$A(x)(y + w) = A(x)(y) + A(x)(w), \quad y, w \in B(0, r/2),$$

i.e. $A(x)$ is additive mapping in the ball $B(0, r/2)$. It is well known that $A(x)$ has the unique extension to an additive map from Y to Z (cf. [1] and [2], Theorem 4.3). Denote this extension by $A(x)$. Setting $\bar{y}_1 = \bar{y}_2 = 0$ in (7) we get

$$|A(x)(y_1) - A(x)(y_2)| \leqq c|y_1 - y_2|, \quad y_1, y_2 \in B(0, r),$$

which implies the continuity of $A(x)$. Since every additive and continuous map is linear we have proved that $A(x) \in L(Y, Z)$. Putting

$$B(x) := h(x, 0), \quad x \in U,$$

we have, according to (8),

$$h(x, y) = A(x)\,y + B(x), \quad x \in U, \quad y \in Y, \quad |y| \leq r,$$

where $A \in F(U, L(Y, Z))$ and $B \in \mathrm{Lip}\,(U, Z)$.

Suppose now that $(Y, |\cdot|)$ is a Banach space. For every $x, \bar{x} \in U$, $x \neq \bar{x}$, we have

$$\frac{A(x) - A(\bar{x})}{|x - \bar{x}|} \in L(Y, Z).$$

From the just proved part of the theorem we have $N(\varphi) - B = A(\cdot)\,y$, for $\varphi(x) = y$. Consequently, for every $y \in B(0, r)$, $A(\cdot)\,y \in \mathrm{Lip}\,(U, Z)$, and, therefore

$$\sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \frac{|A(x)\,y - A(\bar{x})\,y|}{|x - \bar{x}|} = \sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \left| \frac{A(x) - A(\bar{x})}{|x - \bar{x}|}\, y \right| < \infty, \quad y \in B(0, r).$$

This shows that the family of linear maps

$$\left\{ \frac{A(x) - A(\bar{x})}{|x - \bar{x}|} \right\}_{x, \bar{x} \in U, \, x \neq \bar{x}}$$

is pointwise bounded. In view of Banach-Steinhaus Theorem the number

$$\sup_{\substack{x \neq \bar{x} \\ x, \bar{x} \in U}} \frac{\| A(x) - A(\bar{x}) \|}{|x - \bar{x}|}$$

is finite. This completes the proof.

References

[1] Dhombres J., Ger R., Conditional Cauchy Equations, Glasnik Mat. (1978) Vol. 13 (33), p. 39–62.

[2] Dhombres, J., Some Aspects of Functional Equations, Lecture Notes, (1979), Chulalongkorn University, Bangkok.

[3] Matkowski J., Functional Equations and Nemytskii operator, Funkcialaj Ekvacioj 25 (1982), p. 127–132.

[4] Matkowski J., On Nemytskii operator, (will be published in Mathematica Japonica).

[5] Matkowski J., Miś J., On a Characterization of Lipschitzian operators of Substitution in the Space BV⟨a, b⟩, Math. Nachr. 117 (1984), p. 155–159.

[6] Matkowski J., Form of Lipschitz Operators of Substitution in Banach Spaces of Differentiable Functions, Scient. Bull. Łódź Tech. Univ., 17 (1984), p. 5–10.

[7] Matkowska A., On Characterization of Lipschitzian Operators of Substitution in the class of Hölder's Functions, Scient. Bull. Łódź Tech. Univ. 17 (1984), p. 81–85.