FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. 1 (1986), 53—56

EXAMPLES AND REMARKS TO A FIXED POINT THEOREM

Janusz Matkowski and Jerzy Miś

Abstract. A fixed point theorem for nonlinear contractions is under discussion. Three examples and an observation about metric convexity are given.

The first author of this note proved the following fixed point theorem (cf. [3] and J. Dugundji, A. Granas [2], p. 12).

Theorem 1. *Let (X, d) be a complete metric space. Suppose that $F: X \to X$ is a φ-contraction, i.e. that*

$$d(Fx, Fy) \leq \varphi(d(x, y)) \qquad (x, y \in X) \tag{1}$$

where $\varphi: R_+ \to R_+$ satisfies the following two conditions

(i) *φ is non-decreasing not necessarily continuous,*

(ii) *$\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$.*

Then F has a unique fixed point $a \in X$ and $\lim_{n\to\infty} F^n(x) = a$ for each $x \in X$.

(Here and in the sequel $R_+ = [0, \infty)$, φ^n and F^n denote the n-th iterate of φ and F respectively).

This theorem appeared to be useful and easy to handle with in the theory of functional equations (cf. [3]) and therefore it seems to be of interest to answer the question if some of the assumptions (concering φ) can be weaken. In the present note we discuss in detail this question showing in particular, that the answer is no. Example 1 shows that the condition (i) on monotonicity of φ cannot be omitted. One can easily observe that the both conditions (i) and (ii) imply

$$\varphi(t) < t \quad \text{for} \quad t > 0. \tag{iii}$$

This inequality plays an important role in the proof of Theorem 1 (cf. [3] as well as [2], p. 12), therefore one could expect that in remains true after changing (i) by (iii). Our Example 2 settles in negative this a little more delicate problem. In the Example 3 we show that also conditions (i), (ii) cannot be replaced by (i), (iii).

Received July 2, 1986.

On the other hand let us note that if (X, d) is convex in Menger's sense i.e., if for every $x, y \in X$ $(x \neq y)$ there is a $z \in X$ $(z \neq x, z \neq y)$ such that $d(x, z) + d(z, y) = d(x, y)$ (which is usually satisfied in applications), then both the conditions (i) and (ii) can be replaced by (iii) (cf. Boyd-Wong [1]). In this case even much more can be proved. Namely, as a corollary of the main theorem in [4] we have the following result:

Theorem 2. *If (X, d) is complete and convex, $F: X \to X$ satisfies (1) and φ satisfies (iii) then there exists a nondecreasing, concave and continuously differentiable function $\gamma: R_+ \to R_+$ such that $\gamma(t) < t$ for $t > 0$ and*

$$d(Fx, Fy) \leq \gamma(d(x, y)) \quad (x, y \in X).$$

Example 1. Let $X = [0, 1]$, $d(x, y) = |x - y|$ and let Q be the set of rational numbers. The relation „$\sim$" defined as follows

$$x \sim y \Leftrightarrow x - y \in Q$$

is an equivalence relation in X. Let $\{X_\alpha\}_{\alpha \in A}$ be the family of all equivalence classes. Thus

$$X = \bigcup_{\alpha \in A} X_\alpha, \quad X_\alpha \cap X_\beta = \emptyset \quad (\alpha \neq \beta), \quad X_\alpha \neq \emptyset \quad (\alpha \in A).$$

By the axiom of choice there exist two functions

$$A \ni \alpha \mapsto x_\alpha \in X_\alpha, \qquad A \ni \alpha \mapsto y_\alpha \in X - X_\alpha.$$

Now we define the functions $F, \bar{F}: X \to X$ by the formulas

$$F(x) = x_\alpha \text{ iff } x \in X_\alpha; \quad \bar{F}(x) = y_\alpha \text{ iff } x \in X_\alpha,$$

and the function $\varphi: R_+ \to R_+$ by the formula

$$\varphi(t) = \begin{cases} 1, & t \notin Q, \\ t/2, & t \in Q. \end{cases}$$

To check that (1) holds take $x, y \in X$. If $x, y \in X_\alpha$ for some $\alpha \in A$ then, of course, $d(Fx, Fy) = 0$. If there is no such an α, then $d(x, y) \notin Q$ and, consequently, $\varphi(d(x, y)) = 1$. Thus F and φ satisfy (1) i.e., F is a φ-contraction. In the same way we can verify that $\bar{F}$ is a φ-contraction too. Moreover, it follows immediately from the definition of φ that $\varphi^n(t) \to 0$ for each $t > 0$, i.e. condition (ii) holds. Thus, except of (i), all the conditions of of the Theorem 1 are fulfilled. On the other hand one can easily observe that the set of fixed points in the case of mapping F is infinite (equal to $\{x_\alpha\}_{\alpha \in A}$) and in the case of mapping $\bar{F}$ is empty.

Remark 1. Conditions (i) and (ii) does not even assure the uniqueness of fixed point of F.

Example 2. Let $X=\{x_1, x_2, \dots\}$ be an arbitrary countable set. Define $d: X\times X\to R$ as follows

$$d(x_m, x_n)=1+\frac{1}{2^m}+\frac{1}{2^n} \text{ for } n\neq m \text{ and } d(x_n, x_n)=0.$$

Clearly (X, d) is a complete metric space. Note that the metric d has the following property

(2) $$d(x_m, x_n)=d(x_k, x_l)\ (m<n)\wedge k<l) \Leftrightarrow x_m=x_k \wedge x_n=x_l$$

i.e. all the distances $d(x_m, x_n)$ $(m<n)$ are different.

Let $F: X\to X$ be the shifting map $Fx_n=x_{n+1}$. Since the set

$$D=\{d(x_m, x_n): n, m\in N\ (n\neq m)\}$$

is countable and

$$d(Fx_m, Fx_n)=d(x_{m+1}, x_{n+1})<d(x_m, x_n)\quad (n\neq m)$$

we can choose a positive real α_{mn} such that

(3) $$\alpha_{mn}\in(d(x_{m+1}, x_{n+1}), d(x_m, x_n))-D,\ m\neq n\ (m, n\in N).$$

Define now $\varphi: R_+\to R_+$ as follows

$$\varphi(t)=\begin{cases}0, & t\notin D,\\ \alpha_{mn}, & t=d(x_m, x_n)\in D.\end{cases}$$

It follows from the property (2) that φ is correctly defined. By (3) we have $\varphi(t)<t$ for $t>0$ and, because $\varphi^2(t)=0$ for all $t\in R_+$, we also have $\lim\limits_{n\to\infty}\varphi^n(t)=0$. Thus φ satisfies both the conditions (ii) and (iii). Moreover, by (3) and it easily follows that (1) holds, i.e. F is φ-contraction. But clearly F has no fixed point.

Example 3. Let the complete metric space (X, d), D and the map F be as in Example 2. Let use define $\varphi: R_+\to R_+$ as follows

(4) $$\varphi(t)=\begin{cases}t/2, & t\in[0, 1],\\ \sup\{d(x_{m+1}, x_{n+1}): d(x_m, x_n)\leq t;\ m, n\in N\}, & t>1.\end{cases}$$

It follows from the definitions of F and φ that F is a φ-contraction. Evidently, φ is nondecreasing, i.e. φ satisfies condition (i). To see that φ satisfies also (iii) note that every accumulation point t of D such that $t>1$ has the form $t=1+\frac{1}{2^k}$; $k\in N$.

To calculate $\varphi(t)$ take an arbitrary m and n such that

$$d(x_m, x_n)=1+\frac{1}{2^m}+\frac{1}{2^n}\leq t=1+\frac{1}{2^k}.$$

Hence $m \geq k+1$, $n \geq k+1$ and, in view of (4), we have

$$\varphi(t) \leq 1 + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} = 1 + \frac{1}{2^{k+1}} < t.$$

If $t > 1$ is not an accumulation point of the set D then there is a maximal $d(x_m, x_n)$ such that $d(x_m, x_n) \leq t$. Consequently, we have

$$\varphi(t) = d(x_{m+1}, x_{n+1}) < d(x_m, x_n) \leq t.$$

Thus $\varphi(t) < t$ for $t > 0$ which shows that φ satisfies (iii). This completes the required construction.

REFERENCES

1. D. W. Boyd and J. S. Wong: *On nonlinear contractions*. Proc. Amer. Math. Soc. 20 (1969), 458—464.
2. J. Dugundji and A. Granas: *Fixed point theory*. vol. 1, Monografie Matematyczne 61, PWN Warszawa, 1982.
3. J. Matkowski: *Integrable solutions of functional equations*. Dissertationes Mathematicae 127, Warszawa, 1975.
4. J. Matkowski and R. Węgrzyk: *On equivalence of some fixed point theorems for selfmappings of metrically convex space*. Boll. Un. Mat. Ital. A (5) 15 (1978), 359—369.

Technical University
Findera 32
43—300 Bielsko—Biała, Poland

PRIMERI I PRIMEDBE ZA TEOREMU O NEPOKRETNOJ TAČKI

Janusz Matkowski i Jerzy Miś

U radu se razmatra teorema o nepokretnoj tački za nelinearne kontrakcije. Daju se tri primera i jedno opažanje oko konveksnosti metrike.