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Remark on *BV*-solutions of a functional equation connected with invariant measures

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Dedicated to Professor János Aczél on his 60th birthday

We are going to deal with the functional equation

$$\phi(x) = \sum_{i=1}^{k} g_i \phi[f_i(x)], \qquad x \in I = [0, 1], \tag{1}$$

where $f_i: I \to R$ are given functions and g_i are given real numbers. This equation appears when looking for an invariant measure μ defined on the σ-algebra of Borel subsets of I for some mappings $\mathscr{F}: I \to I$. Indeed, suppose $0 = a_0 < a_1 < \ldots < a_k = 1$, $\mathscr{F}_i: [a_{i-1}, a_i] \to I$ is strictly monotonic and onto and define $\mathscr{F}(x) = \mathscr{F}_i(x)$, $x \in [a_{i-1}, a_i)$, $(i = 1, \ldots, k)$, $\mathscr{F}(1) = 1$. If a measure μ is $\mathscr{F}$-invariant, i.e., if $\mu(\mathscr{F}^{-1}(A)) = \mu(A)$ for all Borel sets $A \subset I$ then for $A = [0, x]$ we have

$$\mu([0, x]) = \sum_{i=1}^{k} \mu(\mathscr{F}_i^{-1}[0, x]), \qquad x \in I.$$

Hence, putting $\phi(x) = \mu([0, x])$, $f_i(x) = \mathscr{F}_i^{-1}(x)$, $x \in I$, and observing that $\mu((x, y]) = \phi(y) - \phi(x)$ (in particular $\mu((0, x]) = \phi(x) - \phi(0)$ where $\phi(0)$ is a measure of the one point set $\{0\}$), we get the equation

$$\phi(x) - \phi(0) = \sum_{i=1}^{k} \varepsilon_i[\phi(f_i(x)) - \phi(f_i(0))], \qquad x \in I, \tag{2}$$

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where $\varepsilon_i = 1$ for increasing f_i and $\varepsilon_i = -1$ for decreasing f_i. It is easily seen that equation (2) has form (1) with $n = 2k + 1$ where $f_{k+i}(x) = f_i(0)$, $f_{2k+1}(x) = 0$, $g_i = \varepsilon_i$, $g_{k+i} = -\varepsilon_i$, $g_{2k+1} = 1$, $(i = 1, \ldots, k)$.

Note that equation (2) has the following property: for arbitrary $\phi: [0, 1] \to R$, this equation is satisfied at the points $x = 0$ and $x = 1$. For $x = 0$ is evident and for $x = 1$ we have

$$\sum_{i=1}^{k} \varepsilon_i[\phi(f_i(1)) - \phi(f_i(0))] = \sum_{i=1}^{k} [\phi(a_i) - \phi(a_{i-1})] = \phi(1) - \phi(0).$$

This property of a functional equation is of great importance in our paper and therefore we assume that equation (1) satisfies the following condition: for each function $\phi: [0, 1] \to R$,

$$\phi(0) = \sum_{i=1}^{n} g_i \phi[f_i(0)], \quad \varphi(1) = \sum_{i=1}^{n} g_i \phi[f_i(1)]. \tag{3}$$

REMARK. To obtain another broad class of functional equations of form (1) which satisfy (3), one can take arbitrary $f_i: I \to I$ such that $f_i(0) = 0$, $f_i(1) = 1$, $(i = 1, \ldots, n)$, and $g_i \in R$ such that $g_1 + \ldots + g_n = 1$.

Denote by $BV(I)$ the space of all bounded variation functions $\phi: I \to R$. For $\phi \in BV(I)$ let ϕ^+ and ϕ^- be upper and lower variation of ϕ, respectively. The purpose of this note is to prove the following theorem.

THEOREM. *Let $f_i: I \to I$ be monotonic and suppose that $g_i > 0$ for nondecreasing f_i and $g_i < 0$ for nonincreasing $f_i (i = 1, \ldots, n)$. If, for arbitrary $\phi: I \to R$, conditions (3) are satisfied then for every BV-solution ϕ of eq. (1) the ϕ^+ and ϕ^- satisfy eq. (1).*

Proof. Suppose $\phi \in BV(I)$ is a solution of eq. (1). From the Jordan decomposition theorem, $\phi = \phi^+ - \phi^-$ and for every nondecreasing functions $\phi_1, \phi_2: I \to R$ such that $\phi = \phi_1 - \phi_2$, we have the following two implications

$$0 \leq x \leq y \leq 1 \Rightarrow \phi^+(y) - \phi^+(x) \leq \phi_1(y) - \phi_1(x), \tag{4}$$

$$0 \leq x \leq y \leq 1 \Rightarrow \varphi^-(y) - \phi^-(x) \leq \phi_2(y) - \phi_2(x). \tag{5}$$

Putting in (1) $\phi = \phi^+ - \phi^-$, we get

$$\phi^+(x) - \phi^-(x) = \sum_{i=1}^{n} g_i \phi^+[f_i(x)] - \sum_{i=1}^{n} g_i \phi^-[f_i(x)].$$

It follows from the assumptions of the Theorem that the functions

$$\phi_1 = \sum_{i=1}^{n} g_i \phi^+ \circ f_i, \quad \phi_2 = \sum_{i=1}^{n} g_i \phi^- \circ f_i$$

are nondecreasing and $\phi = \phi_1 - \phi_2$. Hence by (4) we obtain for $0 \leq x \leq y \leq 1$

$$\phi^+(y) - \phi^+(x) \leq \sum_{i=1}^{n} g_i \phi^+[f_i(y)] - \sum_{i=1}^{n} g_i \phi^+[f_i(x)].$$

Putting here $x = 0$, we have by (3)

$$\phi^+(y) \leq \sum_{i=1}^{n} g_i \phi^+[f_i(y)], \quad y \in I.$$

On the other hand, putting $x = 1$ we have

$$\phi^+(x) \geq \sum_{i=1}^{n} g_i \phi^+[f_i(x)], \quad x \in I.$$

The last two inequalities show that ϕ^+ is a solution of eq. (1). In the same way, using (5), we can prove that ϕ^- satisfies eq. (1). This completes the proof.

Note that, if a function f_i is constant, the sign of the corresponding g_i may be arbitrary.

The above theorem reduces the problem of determining all BV-solutions of eq. (1) to the problem of the determining of all monotonic solutions of this equation. But it does not seem to be easy to find all monotonic solutions of eq. (1).

REMARK. Consider the equation

$$\phi(x) = \phi\left(\frac{x}{2}\right) + \phi\left(\frac{x+1}{2}\right) - \phi\left(\frac{1}{2}\right), \quad x \in I. \tag{6}$$

Observe that every linear function $\phi(x) = ax + b$ $(a, b \in R)$ is a monotonic solution of eq. (6). There are also monotonic and discontinuous solutions. Indeed, let us take $\phi_0 \colon \left(\frac{1}{2}, 1\right] \to R$ defined by the formula

$$\phi_0(x) = 2^{-k}, \quad x \in (2^{-1} + 2^{-k-1}, 2^{-1} + 2^{-k}], \quad k = 1, 2, \ldots,$$

and put

$$\phi_n(x) = \phi_{n-1}(2x) - \phi_0\left(\frac{1}{2} + x\right), \qquad x \in (2^{-n-1}, 2^{-n}], \ n = 1, 2, \ldots.$$

Note that the function

$$\phi(x) = \begin{cases} \phi_n(x) & x \in (2^{-n-1}, 2^{-n}], \quad n = 0, 1, \ldots \\ -\dfrac{1}{2} & x = 0 \end{cases}$$

is a solution of eq. (6). Since ϕ_n is nondecreasing and, by a simple calculation, we have

$$\phi_n(x) = \phi_0(2^n x) - \sum_{i=1}^{n} 2^{-i}, \qquad x \in (2^{-n-1}, 2^{-n}],$$

and $\lim_{x \to 0} \phi(x) = -\frac{1}{2}$, it is easily seen that ϕ is nondecreasing. On the other hand it is well known that every absolutely continuous solution of eq. (6) must be a linear function (cf. [2], [1]). Therefore, the following problem seems to be of interest.

PROBLEM. Does the eq. (6) have a nonlinear monotonic and continuous solution?

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