CAUCHY FUNCTIONAL EQUATION ON A RESTRICTED DOMAIN AND COMMUTING FUNCTIONS

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INTRODUCTION

We consider the Cauchy functional equation

$$\phi(x+y) = \phi(x) + \phi(y), \qquad (x,y) \in X \subset R^2, \tag{1}$$

where $\phi: R \to R$ is continuous at least at one point and X is either a union of two perpendicular straight lines x = a, y = b, or a union of two parallel straight lines x = a, x = b (or y = a, y = b), such that a and b are not commensurable. It is easily seen that equation (1) can be written as the following system of two simultaneous functional equations

$$\phi(t+a) = \phi(t) + \phi(a), \quad \phi(t+b) = \phi(t) + \phi(b), \quad t \in R. \tag{2}$$

Theorem 1 says that there is a $c \in R$ such that $\phi(t) = ct$, $t \in R$. This result enables us to prove Theorem 2 which reads as follows. Suppose that $(f^t)_{t \in R}$ is a measurable iteration group of a function $f: I \to I$ which is strictly increasing, onto and without fixed points in the interior of an interval I. If $g: I \to I$ is continuous at least at one point of int(I) and g commutes with two functions f^a and f^b then $g = f^c$ for some $c \in R$.

Using this Theorem we give a contribution to the result of Jens Schwaiger concerning iteration semigroups which are commuting in pairs.

1. WE BEGIN WITH THE FOLLOWING

Theorem 1. Suppose that $\phi: R \to R$ satisfies (2) where $a, b \in R$, $a \neq 0$, and b/a is irrational. If ϕ is continuous at least at one point then there is a $c \in R$ such that $\phi(t) = ct$ for $t \in R$.

Proof. Suppose first that ϕ is continuous at t = 0. It follows easily from (2) that

$$\phi(t+na) = \phi(t)+n\phi(a), \; \phi(t+mb) = \phi(t)+m\phi(b), \quad n,m \in Z, \; t \in R, \tag{3}$$

which implies that

$$\phi(t+na+mb) = \phi(t)+n\phi(a)+m\phi(b), \quad n,m \in Z, \quad t \in R. \tag{4}$$

Because $\varphi(0) = 0$ we hence get

$$\varphi(na+mb) = n\varphi(a)+m\varphi(b), \quad n,m \in \mathbb{Z}. \tag{5}$$

The set

$$A = \{na+mb: n,m \in \mathbb{Z}\}$$

is dense in R (cf. Halmos [1], p.69, Theorem 6). Therefore there exists a sequence $t_k = n_k a + m_k b \in A$, $k = 1,2,\dots$, such that

$$\lim_{k\to\infty} t_k = 0, \quad t_k = n_k a + m_k b \neq 0, \quad k \in \mathbb{N}. \tag{6}$$

From (6) by the continuity of φ at the point $t = 0$ we have

$$\lim_{k\to\infty} (n_k\varphi(a)+m_k\varphi(b)) = \varphi(0) = 0 . \tag{7}$$

It follows from (6) that $m_k \neq 0$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \frac{n_k}{m_k} = -\frac{b}{a} .$$

Now (7) implies that

$$\lim_{k\to\infty} \frac{n_k\varphi(a)+m_k\varphi(b)}{m_k} = -\frac{b}{a}\varphi(a)+\varphi(b) = 0 ,$$

i.e., $\frac{\varphi(a)}{a} = \frac{\varphi(b)}{b}$. Consequently, there exists a $c \in \mathbb{R}$ such that

$$\varphi(a) = ca, \quad \varphi(b) = cb.$$

Hence and from (5) we obtain $\varphi(na+mb) = c(na+mb)$ for $n,m \in \mathbb{Z}$, i.e.

$$\varphi(t) = ct, \quad t \in A.$$

Take $t_0 \in \mathbb{R}$. Since the set t_0+A is dense in R there exists a sequence $t_k \in A$, $t_k = n_k a + m_k b$, $(k \in \mathbb{N})$, such that

$$\lim_{k\to\infty} (t_0+t_k) = 0, \quad t_k \neq 0, \quad k \in \mathbb{N}.$$

Now, according to (4), we have

$$\phi(t_0+n_k a+m_k b) = \phi(t_0)+c(n_k a+m_k b) = c(t_0+n_k a+m_k b)+(\phi(t_0)-ct_0)$$

and the continuity of ϕ at the point $t = 0$ implies that $\phi(t_0) = ct_0$. This proves that $\phi(t) = ct$ for all $t \in R$.

To finish the proof note that if ϕ is continuous at a point $t_0 \neq 0$ and satisfies equations (2) then the function $\psi: R \to R$ defined by the formula $\psi(t) = \phi(t+t_0) - \phi(t_0)$ is continuous at the point $t = 0$ and also satisfies equations (2).

Remark. The same reasoning as in the above proof and observation that $m_k \to \infty$ as $k \to \infty$ allows us to prove that if ϕ is bounded in a neighbourhood of $t = 0$ then $\phi(t) = ct$ for all t belonging to the dense set A. But the boundedness of ϕ in a neighbourhood of $t = 0$ does not imply that $\phi(t) = ct$ for $t \in R$. In fact, $\phi(t) = 0$ for $t \in A$ and $\phi(t) = 1$ for $t \in R-A$ satisfies eq.(2), is bounded and, evidently, it is not linear.

As a simple consequence of Theorem 1 we get the following useful.

Corollary 1. Let $a,b,\alpha,\beta \in R$ and suppose that $\varphi: R \to R$ is continuous at least at one point and satisfies the following system of equations

$$\varphi(t+a) = \varphi(t)+\alpha, \quad \varphi(t+b) = \varphi(t) + \beta . \tag{8}$$

(a) If $a \neq 0$ and b/a is irrational then there exist $c_0, c_1 \in R$ such that $\varphi(t) = c_0 + c_1 t$ for $t \in R$.

(b) If $\alpha \neq 0$ and β/α is irrational then $a \neq 0$ and b/a is irrational.

Proof. (a) Put $c_0 = \varphi(0)$ and note that $\phi(t) = \varphi(t) - \varphi(0)$ satisfies equations (2).

(b) From (8) we have $\varphi(na+mb) = \varphi(0)+n\alpha+m\beta$, $n,m \in Z$. Now $\alpha \neq 0$ implies $a \neq 0$. If b/a were rational we would find integers n and m, at least one different from 0, such that $na+mb = 0$. Consequently, we would have $\varphi(0) = \varphi(na+mb) = \varphi(0) + n\alpha + m\beta$.

Hence we conclude that $n\alpha+m\beta = 0$ which is a contradiction. For the multiplicative version of the Cauchy equation we get in the same way

Corollary 2. Let a,b,α,β be positive real and suppose that $\psi: (0,\infty) \to (0,\infty)$ is continuous at least at one point and satisfies the following system of simultaneous Schröder functional equations

$$\psi(at) = \alpha\psi(t), \quad \psi(bt) = \beta\psi(t).$$

(a) If $a \neq 1$ and $\ln b/\ln a$ is irrational then there are $c_0 > 0$ and $c_1 \in R$ such that $\psi(t) = c_0 t^{c_1}$.

(b) If $\alpha \neq 1$ and $\ln\beta/\ln\alpha$ is irrational then $a \neq 1$ and $\ln b/\ln a$ is irrational.

Remark. Corollary 2 is a generalization of a similar result in [2] where φ is assumed to be monotonic. This result can be applied to a characterization of L^p norm.

2. IN THIS SECTION WE CONSIDER ITERATION GROUPS AND COMMUTING FUNCTIONS

Let I be an interval. A family of functions $(f^t)_{t \in R}$ is said to be an iteration group of a function $f: I \to I$ iff $f^t: I \to I$, $f^1 = f$, $f^t \circ f^s = f^{t+s}$ for all $t,s \in R$. Replacing in this definition R by the interval $(0,\infty)$ we obtain the definition of an iteration semigroup.

An iteration group $(f^t)_{t \in R}$ (semigroup $(f^t)_{t > 0}$) is said to be continous resp. measurable if for every $x \in I$ the mapping $R \ni t \to f^t(x)$ ($(0,\infty) \ni t \to f^t(x)$) is continous resp. measurable.

Now we are going to prove the following

Theorem 2. Let $f: I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t \in R}$ is a measurable iteration group of f with each f^t continuous. If a function $g: I \to I$ continuous at least at one point commutes with two functions f^a and f^b such that b/a is irrational then there exists a $c \in R$ such that $g = f^c$ i.e., g belongs to the iteration group of f.

Proof. According to Zdun's theorem, (f^t) has to be a continous iteration group (cf.[3]). Therefore there exists a function $\alpha: R \to I$ continous, strictly monotonic and onto such that

$$f^t(x) = \alpha(t+\alpha^{-1}(x)), \quad x \in I, \quad t \in R. \tag{9}$$

Using the assumption $f^a \circ g = g \circ f^a$ we have

$$\alpha(a+\alpha^{-1}(g(x))) = g(\alpha(a+\alpha^{-1}(x))), \quad x \in I.$$

Since α is onto, for each $t \in R$ there is an $x \in I$ such that $x = \alpha(t)$. Substituting this into the above equation we get

$$\alpha(a+\alpha^{-1}(g(\alpha(t)))) = g(\alpha(a+t)), \quad t \in R.$$

Hence, putting

$$\phi = \alpha^{-1} \circ g \circ \alpha \tag{10}$$

we get the equation

$$\varphi(t+a) = \varphi(t)+a \ . \tag{11}$$

Similarly, from $f^b \circ g = g \circ f^b$ we have

$$\varphi(t+b) = \varphi(t)+b \ . \tag{12}$$

Since φ is continuous at least at one point, in view of Corollary 1 with $\alpha = a$ and $\beta = b$ we obtain $\varphi(t) = c_0 + c_1 t$ for $t \in R$. From each of equations (11) - (12) it follows that $c_1 = 1$, i.e., $\varphi(t) = c+t$, $t \in R$, where $c = c_0$. According to (10) we now have $g(\alpha(t)) = \alpha(c+t)$. Setting here $t = \alpha^{-1}(x)$ we have

$$g(x) = \alpha(c+\alpha^{-1})(x)), \quad x \in I,$$

which in view of (9) means that $g = f^c$. This completes the proof.

Using Theorem 2 we can prove the following

<u>Theorem 3.</u> Suppose that $f: I \to I$ is strictly increasing, onto and without fixed points in an open interval I. Let $(f^t)_{t \in R}$ be a measurable iteration group of f such that for each $t \in R$, f^t is continuous in I. Let $(g^t)_{t>0}$ be a continuous iteration semigroup such that for each $t>0$, g^t is continuous at least at one point. If

$$f^t \circ g^t = g^t \circ f^t, \quad t > 0, \tag{13}$$

then there is a $c \in R$ such that $g^t = f^{ct}$ for all $t > 0$.

<u>Proof.</u> From (13) we have $f^{nt} \circ g^{mt} = g^{mt} \circ f^{nt}$ for $t > 0$, $n,m \in N$. Setting here $t = s/n$ we get $f^s \circ g^{ws} = g^{ws} \circ f^s$ for all $s > 0$ and all positive rationals w. Take positive reals s and t and a sequence (w_n) of positive rationals such that

$$\lim_{n\to\infty} w_n = \frac{t}{s} \ .$$

Thus we have

$$f^s \circ g^{w_n s} = g^{w_n s} \circ f^s, \quad n \in N.$$

Since f^s is continuous and $(g^t)_{t>0}$ is a continuous iteration semigroup it follows that

$$f^s \circ g^t = g^t \circ f^s, \quad s,t > 0.$$

In view of Theorem 2 for each $t > 0$ there is a $C(t) \in R$ such that

$$g^t = f^{C(t)} .$$

Now we have

$$f^{C(t+s)} = g^{t+s} = g^t \circ g^s = f^{C(t)+C(s)} , \quad t,s > 0 .$$

Since (f^t) is strictly monotonic with respect to t (cf. formula (9)), it follows that $C(t+s) = C(t)+C(s)$ for $t,s > 0$. Let us take $t_n, t_0 > 0$ such that $\lim_{n\to\infty} t_n = t_0$. The continuity of (g^t) with respect to t implies that $\lim_{n\to\infty} g^{t_n} = g^{t_0}$, which means that $\lim_{n\to\infty} f^{C(t_n)} = f^{C(t_0)}$. Making use once more of the strict monotonicity of (f^t) with respect to t, we obtain $\lim_{n\to\infty} C(t_n) = C(t_0)$. Thus $C:(0,\infty) \to R$ is additive and continuous. Consequently, there is a constant $c \in R$ such that $C(t) = ct$ for $t > 0$. This completes the proof.

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