Math. Nachr. 117 (1984) 153–159

On a Characterization of Lipschitzian Operators of Substitution in the Space $BV\langle a, b\rangle$

By J. Matkowski and J. Miś of Bielsko-Biała

(Received October 12, 1982)

Let $BV\langle a, b\rangle$ be the Banach space of functions of bounded variation $\varphi : \langle a, b\rangle \to R$ with the norm

$$\|\varphi\| = |\varphi(a)| + \bigvee_a^b \varphi\,, \tag{1}$$

where $\bigvee_a^b \varphi$ denotes the total variation of φ over the interval a, b, i.e.,

$$\bigvee_a^b \varphi = \sup \sum_{i=1}^{m} |\varphi(x_i) - \varphi(x_{i-1})|\,, \tag{2}$$

where supremum is taken over all positive integers m and over all choices $\{x_i\}$ such that $a = x_0 < x_1 < \ldots < x_m = b$.

Denote by $F\langle a, b\rangle$ the vector space of all functions $\varphi : \langle a, b\rangle \to R$. Every function

$$h : \langle a, b\rangle \times R \to R$$

generates the so-called Nemytskii operator or operator of substitution $N = N_h : F\langle a, b\rangle \to F\langle a, b\rangle$ defined by the formula

$$(N\varphi)(x) = h(x, \varphi(x)), \quad (x \in \langle a, b\rangle)\,. \tag{3}$$

Now we can prove the following

Theorem 1. *Suppose that $h : \langle a, b\rangle \times R \to R$. If the operator N defined by (3) fulfils the following conditions*

$$N : BV\langle a, b\rangle \to BV\langle a, b\rangle\,, \tag{4}$$

and N is a Lipschitz map in the sense of norm (1), i.e., there is an $L > 0$ such that

$$\|N\varphi - N\bar{\varphi}\| \le L\,\|\varphi - \bar{\varphi}\|, \quad (\varphi, \bar{\varphi} \in BV\langle a, b\rangle) \tag{5}$$

then

a) $|h(x, y) - h(x, \bar{y})| \le L\,|y - \bar{y}|$, $(x \in \langle a, b\rangle;\ y, \bar{y} \in R)$;

b) *for every $x \in (a, b\rangle$ there exists $h^*(x, y) = \lim_{t \to x^-} h(t, y)$, $y \in R$ and there exist left-continuous functions G, $H \in BV\langle a, b\rangle$ such that*

$$h^*(x, y) = G(x)\,y + H(x), \quad x \in (a, b\rangle, \quad y \in R\,. \tag{6}$$

Proof. For fixed $x\in(a,b)$ and $y,\bar{y}\in R$ we define $\varphi,\psi:(a,b)\to R$ as follows

$$\varphi(t)=y,\qquad t\in(a,b);\qquad \psi(t)=\begin{cases} y, & t\neq x \\ \bar{y}, & t=x\end{cases}.$$

It is easily seen that $\varphi,\psi\in BV(a,b)$ and in the case $x\in(a,b)$ we have

$$|\varphi-\psi|=2\,|y-\bar{y}|;\qquad |N\varphi-N\psi|=2\,|h(x,y)-h(x,\bar{y})|,$$

and in the case $x=b$ we have

$$|\varphi-\psi|=|y-\bar{y}|;\qquad |N\varphi-N\psi|=|h(x,y)-h(x,\bar{y})|.$$

Thus in the both cases condition (5) implies that $|h(x,y)-h(x,\bar{y})|\le L\,|y-\bar{y}|$ which completes the proof of a).

b). Since for every fixed $y\in R$ the function $\varphi(t)=y$ for $t\in(a,b)$ as a constant is of bounded variation, it follows from (4) that $N\varphi=h(\cdot,y)\in BV(a,b)$. Hence $h(\cdot,y)$ has the one-sided limits at every point $x\in(a,b)$ and, consequently, the function h^* exists. Note that for every $y\in R$, $h^*(\cdot,y)$ is left-continuous and $h^*(\cdot,y)\in BV(a,b)$.

Let us fix $y_1,\bar{y}_1,y_2,\bar{y}_2\in R$, $x\in(a,b)$ and choose a positive integer n and $x_1,x_2,\dots,x_{2n}$ such that $a<x_1<x_2<\dots<x_{2n}<x$. Define now $\varphi,\psi:(a,b)\to R$ as follows

$$\varphi(t)=\begin{cases} y_1, & t\notin\{x_2,x_4,\dots,x_{2n}\} \\ y_2, & t\in\{x_2,x_4,\dots,x_{2n}\}\end{cases};$$

$$\psi(t)=\begin{cases} \bar{y}_1, & t\notin\{x_2,x_4,\dots,x_{2n}\} \\ \bar{y}_2, & t\in\{x_2,x_4,\dots,x_{2n}\}\end{cases}.$$

It is easily seen that $\varphi,\psi\in BV(a,b)$ and

$$|\varphi-\psi|=|y_1-\bar{y}_1|+2n\,|y_2-\bar{y}_2-y_1+\bar{y}_1|. \tag{7}$$

From (2) we have

$$\sum_{i=1}^{n}|h(x_{2i},y_2)-h(x_{2i},\bar{y}_2)-h(x_{2i-1},y_1)+h(x_{2i-1},\bar{y}_1)|\le\bigvee_a^b(N\varphi-N\psi). \tag{8}$$

It follows from (1) and (5) that

$$\bigvee_a^b(N\varphi-N\psi)\le|N\varphi-N\psi|\le L\,|\varphi-\psi|.$$

Hence by (7) and (8) we obtain the following inequality

$$\sum_{i=1}^{n}|h(x_{2i},y_2)-h(x_{2i},\bar{y}_2)-h(x_{2i-1},y_1)+h(x_{2i-1},\bar{y}_1)|$$
$$\le L\,(|y_1-\bar{y}_1|+2n\,|y_2-\bar{y}_2-y_1+\bar{y}_1|)$$

for all $x\in(a,b)$, $y_1,\bar{y}_1,y_2,\bar{y}_2\in R$ and $a<x_1<x_2<\dots<x_{2n}<x$. Letting in this inequality $x_1\to x$ we get

$$n\,|h^*(x,y_2)-h^*(x,\bar{y}_2)-h^*(x,y_1)+h^*(x,\bar{y}_1)|$$
$$\le L(|y_1-\bar{y}_1|+2n|\,y_2-\bar{y}_2-y_1+\bar{y}_1|)$$

for all $x\in\langle a,b)$ and $y_1,\bar{y}_1,y_2,\bar{y}_2\in R$. Setting $\bar{y}_1=0$, $y_1=v$, $\bar{y}_2=u$, $y_2=u+v$ in the above inequality we have

$$n\,|h^*(x,u+v)-h^*(x,u)-h^*(x,v)+h^*(x,0)|\le L\,|v|$$

for $x\in\langle a,b)$; $u,v\in R$. Hence, letting $n\to\infty$, we conclude that

$$h^*(x,u+v)-h^*(x,u)-h^*(x,v)+h^*(x,0)=0,\quad u,v\in R.$$

This equation may be written in the following form

(9) $$[h^*(x,u+v)-H(x)]=[h^*(x,u)-H(x)]+[h^*(x,v)-H(x)],\quad u,v\in R$$

where

$$H(x)=h^*(x,0),\quad x\in\langle a,b\rangle.$$

Let us fix an $x\in\langle a,b)$ and define the function $f_x:R\to R$ by the formula

(10) $$f_x(y)=h^*(x,y)-H(x),\quad y\in R.$$

Hence by (9) we have

$$f_x(u+v)=f_x(u)+f_x(v),\quad u,v\in R,$$

i.e., f_x is additive. By a) and the definition of h^* we have

$$|h^*(x,y)-h^*(x,\bar{y})|\le L\,|y-\bar{y}|,\quad x\in\langle a,b);\ y,\bar{y}\in R,$$

which implies the continuity of f_x. Therefore there exists $G(x)\in R$ such that

$$f_x(y)=G(x)\,y,\quad y\in R,$$

and by (10) we get

$$h^*(x,y)=G(x)\,y+H(x),\quad x\in\langle a,b);\ y\in R.$$

Since $H=h^*(\cdot,0)$ and $G=h^*(\cdot,1)-h^*(\cdot,0)$, the functions G and H are left-continuous and $G,H\in BV\langle a,b\rangle$. This completes the proof.

Consider now the following

Example. Take $\langle a,b\rangle\subseteq R$ and the sequence (w_k) of all rational numbers of the interval $\langle a,b\rangle$ such that $w_i\neq w_k$ for $i\neq k$. Define

$$h(x,y)=\begin{cases}0, & x\neq w_k;\quad y\in R\\ 2^{-k}\sin y, & x=w_k;\quad y\in R\end{cases}.$$

For $\varphi\in BV\langle a,b\rangle$ and for (x_i) such that $a=x_0<x_1<\ldots<x_n=b$ we have

$$\sum_{i=1}^{n}|h(x_i,\varphi(x_i))-h(x_{i-1},\varphi(x_{i-1}))|\le 2\sum_{i=0}^{n}|h(x_i,\varphi(x_i))|$$

$$\le 2\sum_{k=1}^{\infty}|h(w_k,\varphi(w_k))|=2\sum_{k=1}^{\infty}2^{-k}|\sin\varphi(w_k)|\le 2\sum_{k=1}^{\infty}2^{-k}=2,$$

which shows that $N=N_h: BV\langle a,b\rangle\to BV\langle a,b\rangle$.

Now for $\varphi,\bar{\varphi}\in BV\langle a,b\rangle$ and for (x_i) such that $a=x_0<x_1<\ldots<x_n=b$ we have

$$\sum_{i=1}^{n}|(N\varphi-N\bar{\varphi})(x_i)-(N\varphi-N\bar{\varphi})(x_{i-1})|$$

$$=\sum_{i=1}^{n}|h(x_i,\varphi(x_i))-h(x_i,\bar{\varphi}(x_i))-h(x_{i-1},\varphi(x_{i-1}))+h(x_{i-1},\bar{\varphi}(x_{i-1}))|$$

$$\le 2\sum_{i=1}^{n} |h(x_i, \varphi(x_i)) - h(x_i, \bar\varphi(x_i))| \le 2\sum_{k=1}^{\infty} |h(w_k, \varphi(w_k)) - h(w_k, \bar\varphi(w_k))|$$

$$= 2\sum_{k=1}^{\infty} 2^{-k} |\sin \varphi(w_k) - \sin \bar\varphi(w_k)| \le 2\sum_{k=1}^{\infty} 2^{-k} |\varphi(w_k) - \bar\varphi(w_k)|$$

$$= 2\sum_{k=1}^{\infty} 2^{-k} |(\varphi - \bar\varphi)(w_k)| \le 2\sum_{k=1}^{\infty} 2^{-k} \|\varphi - \bar\varphi\| = 2\|\varphi - \bar\varphi\| .$$

Hence $\bigvee_a^b (N\varphi - N\bar\varphi) \le 2\|\varphi - \bar\varphi\|$ and, consequently, we have

$$\|N\varphi - N\bar\varphi\| = |(N\varphi - N\bar\varphi)(a)| + \bigvee_a^b (N\varphi - N\bar\varphi) \le |\sin \varphi(a) - \sin \bar\varphi(a)|$$
$$+ 2\|\varphi - \bar\varphi\| \le |\varphi(a) - \bar\varphi(a)| + 2\|\varphi - \bar\varphi\| \le 3\|\varphi - \bar\varphi\| .$$

Thus N generated by h is a Lipschitzian selfmapping of $BV(a, b)$. By Theorem 1 the left regularization $h^* \equiv 0$ of the function h has form (6). This example shows, in particular, that the function h in Theorem 1 need not be linear with respect to the second variable.

Imposing an additional condition on the left regularization h^* of h we have the following more satisfactory result.

Theorem 2. *Suppose that $h : (a, b) \times R \to R$, there exists h^* and $h = h^*$. Then for $N = N_h$ the following two conditions are equivalent:*

(i) *$N : BV(a, b) \to BV(a, b)$ is Lipschitzian;*

(ii) *there exist the left-continuous functions G, $H \in BV(a, b)$ such that $h(x, y) = G(x)\, y + H(x)$ for $x \in (a, b)$, $y \in R$.*

Proof. By Theorem 1 it is sufficient to prove the implication (ii) ⇒ (i). Let $N = N_h$ where h is given in (ii). Since G, $H \in BV(a, b)$, it is clear that $N : BV(a, b) \to BV(a, b)$. Now take φ, $\bar\varphi \in BV(a, b)$ and put $\psi = \varphi - \bar\varphi$. Hence by the definition of the norm (1) we have

$$\|N\bar\varphi - N\varphi\| = \|G\psi\| = |G(a)\, \psi(a)| + \sup \sum_{i=1}^{n} |G(x_i)\, \psi(x_i) - G(x_{i-1})\, \psi(x_{i-1})|$$

$$\le |G(a)\, \psi(a)| + \sup \sum_{i=1}^{n} (|G(x_i)|\, |\psi(x_i) - \psi(x_{i-1})| + |\psi(x_{i-1})|\, |G(x_i) - G(x_{i-1})|)$$

$$\le (\sup_{[a,b]} |G|)\, |\psi(a)| + (\sup_{[a,b]} |G|) \bigvee_a^b \psi + \left(\bigvee_a^b G\right) \sup_{[a,b]} |\psi|$$

$$\le (\sup_{[a,b]} |G|) \left(|\psi(a)| + \bigvee_a^b \psi\right) + \left(\bigvee_a^b G\right)\left(|\psi(a)| + \bigvee_a^b \psi\right)$$

$$= \left(\sup |G| + \bigvee_a^b G\right) \|\psi\| = \left(\sup |G| + \bigvee_a^b G\right) \|\varphi - \bar\varphi\| .$$

This completes the proof.

Remark 1. One can easily observe that similar theorems hold for the right regularization of h. Moreover, the interval (a, b) may be replaced by any interval (a, b), where $-\infty \le a < b \le \infty$.

Remark 2. Our theorems show that a bounded variation solution of the nonlinear functional equation $\varphi(x)=h(x, \varphi[f(x)])$ cannot be obtained by a direct application of the BANACH's fixed point theorem (cf. [3]).

Remark 3. Note that the condition of compactness of the NEMYTSKII operator $N: L^p(a,b)\to L^p(a,b)$ is very strong too. KRASNOSELSKII [1], p. 30 (see also [2], p. 163) has proved that every such operator is a constant map.

References

[1] M. A. KRASNOSELSKII, Topological Methods in the Theory of Nonlinear Integral Equations, MacMillan, New York 1964

[2] R. Jr. MARTIN, Nonlinear Operators in Banach Spaces, John Wiley & Sons Inc., New York 1976

[3] J. MATKOWSKI, Functional equations and Nemytskii Operator, Proc. 18-th Int. Symp. on Funct. Eq., Waterloo, Canada 1980, p. 19–20

Lubertowicza 2/1
PL-43-380 Bielsko-Biała
Poland

Wiśniowa 10/17
PL-43-380 Bielsko-Biała
Poland