ANNALES
POLONICI MATHEMATICI
XXIII (1970)

On the continuous dependence of solutions of a functional equation on given functions

by J. Matkowski (Katowice)

The subject of the present paper is the problem of the continuous dependence of analytic solutions of the functional equations

$$\varphi[f(z)] - g(z)\varphi(z) = h(z), \tag{1}$$

where f, g, h are given functions, φ is unknown and z is a complex variable.

Together with (1) we shall consider a sequence of the equations

$$\varphi[f_n(z)] - g_n(z)\varphi(z) = h_n(z). \tag{2}$$

We shall assume that

(I) f_n is analytic in the domain U; $f_n(U) \subset U$; the boundary of U contains at least two points; $0 \in U$; $f_n(z) = z \Leftrightarrow z = 0$, and $|f_n'(0)| \leq \vartheta < 1$, $n = 1, 2, 3, \ldots$

(II) g_n and h_n are analytic functions in U, $g_n(z) \neq 0$ for $z \in U$; $g_n(0) \neq 1$ and $g_n(0) \neq [f_n'(0)]^k$ for $n, k = 1, 2, \ldots$

It follows from W. Smajdor's theorem [3] (see also [1], pp. 187-191) and Lemma 2 in [2] that under hypotheses (I), (II), equation (2) has exactly one analytic solution φ_n in U.

Suppose that

(III) f_n, g_n, h_n tend uniformly on the every compact $K \subset U$ to f, g, h; accordingly, $g(z) \neq 0$ for $z \in U$ and $g(0) \neq 1$, $g(0) \neq [f'(0)]^k$ for $k = 1, 2, \ldots$

The function f satisfies (I)(¹) and the functions g and h satisfy (II); thus equation (1) has the unique analytic solution φ in U.

We shall prove the following

THEOREM. *Under hypotheses* (I)-(III) $\{\varphi_n\}$ *tends to* φ *uniformly on every compact* $K \subset U$.

For the proof we need some lemmas.

(¹) In particular, the relation $f(z) = z \Leftrightarrow z = 0$ results from Hurwitz's theorem owing to the inequality postulated above in (I) for $f_n'(0)$.

LEMMA 1. *If the function f fulfils hypothesis* (I), *then the sequence* $\{f^n\}$ *of the iterates of f tends to* 0 *uniformly on every compact* $K \subset U$.

The proof of this lemma may be found in [2].

LEMMA 2. $\varphi_n^{(k)}(0)$ *tends to* $\varphi^{(k)}(0)$ *as* $n \to \infty$ *for* $k = 1, 2, 3, \ldots$

Proof. Since

$$\varphi_n(0) = \frac{h_n(0)}{1-g_n(0)} \quad \text{tends to} \quad \frac{h(0)}{1-g(0)} = \varphi(0),$$

Lemma 2 holds for $k = 0$. Writing φ_n in place of φ in equation (2) and differentiating k times, we obtain

$$\varphi_n^{(k)}[f_n(z)][f_n'(z)]^k + \sum_{i=0}^{k-1} \varphi_n^{(i)}(z)\, W_i^{(k)}[f_n'(z), \ldots, f_n^{(k)}(z)] - \tag{3}$$
$$- \sum_{i=0}^{k} \binom{k}{i} \varphi_n^{(i)}(z) g_n^{(k-i)}(z) = h_n^{(k)}(z),$$

where $W_i^{(k)}(w_1, \ldots, w_k)$ is a polynomial in k variables (independent of n). Putting $z = 0$ in (3), we get

$$\varphi_n^{(k)}(0) \tag{4}$$
$$= \frac{h_n^{(k)}(0) - \sum_{i=1}^{k-1} \varphi_n^{(i)}(0)\, W_i^{(k)}[f_n'(0), \ldots, f_n^{(k)}(0)] + \sum_{i=0}^{k-1} \binom{k}{i} \varphi_n^{(i)}(0) g_n^{(k-i)}(0)}{[f_n'(0)]^k - g_n(0)}.$$

Similary we obtain

$$\varphi^{(k)}(0) \tag{5}$$
$$= \frac{h^{(k)}(0) - \sum_{i=1}^{k-1} \varphi^{(i)}(0)\, W_i^{(k)}[f'(0), \ldots, f^{(k)}(0)] + \sum_{i=0}^{k-1} \binom{k}{i} \varphi^{(i)}(0) g^{(k-i)}(0)}{[f'(0)]^k - g(0)}.$$

Suppose that $\varphi_n^{(i)}(0)$ tends to $\varphi^{(i)}(0)$ as $n \to \infty$ for $i = 1, 2, \ldots, k-1$. Then from (4) and (5) we see that

$$\varphi_n^{(k)}(0) \xrightarrow[n\to\infty]{} \varphi^{(k)}(0)$$

and induction completes the proof of Lemma 1.

LEMMA 3. *The sequence* $\{\varphi_n\}$ *of solutions of equation* (2) *forms a normal family in* U.

Proof. It follows from (I), (II) and (III) that there exist $A > 0$ and $r > 0$ such that

$$|f_n^p(z)| \leqslant \theta^p |z| \quad \text{for } n, p = 1, 2, 3, \ldots, \text{ and } |z| \leqslant r, \tag{6}$$

where $0 < \vartheta < 1$;

(7) $$|g_n(z)| \geqslant A \quad \text{for } n = 1, 2, 3, \ldots, \text{ and } |z| \leqslant r.$$

Let us choose a positive integer m such that

(8) $$\vartheta^m < A.$$

Evidently every analytic solution of equation (2) may be written in the form

(9) $$\varphi_n(z) = P_n(z) + \Phi_n(z),$$

where

$$P_n(z) = \sum_{i=0}^{m-1} \frac{\varphi_n^{(i)}(0)}{i!} z^i$$

and $\Phi_n(z)$ is analytic in U.

It is easy to verify that $\Phi_n(z)$ is an analytic solution of the equation

(10) $$\Phi[f_n(z)] - g_n(z)\Phi(z) = h_n^*(z),$$

where

$$h_n^*(z) = h_n(z) + g_n(z)P_n(z) - P_n[f_n(z)]$$

and $h_n^*(z)$ may be written as $h_n^*(z) = z^m H_n^*(z)$, where $H_n^*(z)$ is analytic in U. The last relation implies that there exists a $B > 0$ such that

(11) $$|h_n^*(z)| \leqslant B|z|^m \quad \text{for } |z| \leqslant r; n = 1, 2, 3, \ldots$$

It is known (see [1], pp. 52-53) that the analytic solution of equation (10) may be written in the form

(12) $$\Phi_n(z) = -\sum_{p=0}^{\infty} \frac{h_n^*[f_n^p(z)]}{\prod_{i=0}^{p} g_n[f_n^i(z)]}.$$

For $|z| \leqslant r$ we have $|f_n^i(z)| \leqslant \vartheta^i |z| < r$, and thus we get by (12) and (6), (7), (11), (8),

$$|\Phi_n(z)| \leqslant \sum_{p=0}^{\infty} \frac{B|f_n^p(z)|^m}{\prod_{i=0}^{p} |g_n[f_n^i(z)]|} \leqslant \sum_{p=0}^{\infty} \frac{B\vartheta^{pm}|z|^m}{A^{p+1}} \leqslant \frac{Br^m}{A} \sum_{p=0}^{\infty} \left(\frac{\vartheta^m}{A}\right)^p = M.$$

Thus $\{\Phi_n(z)\}$ is a normal family in the disc $|z| < r$. From Lemma 2 we find that $P_n(z)$ tends to

$$P(z) = \sum_{i=0}^{m-1} \frac{\varphi^{(i)}(0)}{i!} z^i \quad \text{as } n \to \infty$$

uniformly for $|z| \leqslant r$, and so also $\{\varphi_n(z)\}$ is a normal family for $|z| < r$.

Now we denote by V the maximal set of normality of the sequence $\{\varphi_n(z)\}$. Evidently V is open. Suppose that $U \setminus V \neq \emptyset$. Then it follows from Lemma 1 that there exists a $z_0 \in U \setminus V$ such that $f(z_0) \in V$. Hence and from (I) it follows that there exist a neighbourhood U_{z_0} of z_0 and an integer N such that for $n \geq N$ we have $f_n(U_{z_0}) \subset V$. Now we conclude that the sequence $\{\varphi_n\}$ is normal at the point z_0, for $\{\varphi_n[f_n(z)]\}$ is normal at z_0 and

$$\varphi_n(z) = \frac{\varphi_n[f_n(z)] - h_n(z)}{g_n(z)}.$$

Thus $U \setminus V = \emptyset$ and Lemma 3 is proved.

Proof of the Theorem. Suppose that the theorem is false. It follows from Lemma 3 that we can choose a subsequence $\{\varphi_{n_k}\}$ uniformly convergent on every compact $K \subset U$ to $\psi \neq \varphi$. Passing to the limit in the relation

$$\varphi_{n_k}[f_{n_k}(z)] - g_{n_k}(z)\varphi_{n_k}(z) = h_{n_k}(z),$$

we get $\psi[f(z)] - g(z)\psi(z) = h(z)$. Since ψ is an analytic solution of (1), we must have $\psi = \varphi$. This contradiction completes the proof.

References

[1] M. Kuczma, *Functional equations in a single variable*, Warszawa 1968.

[2] J. Matkowski, *On meromorphic solutions of a functional equation*, Ann. Polon. Math. (to appear).

[3] W. Smajdor, *On the existence and uniqueness of analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$, ibidem 19 (1967), pp. 37-45.

Reçu par la Rédaction le 14. 7. 1968