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FORM OF LIPSCHITZ OPERATORS OF SUBSTITUTION IN BANACH SPACES OF DIFFERENTIABLE FUNCTIONS

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Let $C^n\langle a,b\rangle$ be the Banach space of n-times differentiable functions $\varphi: \langle a,b\rangle \to R$ with the usual norm $\|\varphi\|_n$ and let $h: \langle a,b\rangle \times R \to R$ be of class C^n in $\langle a,b\rangle \times R$. In particular, it is shown that every substitution operator $H: C^n\langle a,b\rangle \to C^m\langle a,b\rangle$, $(m<n)$, defined by the formula $(H\varphi)(x) = h(x, \varphi(x))$ which fulfils the Lipschitz condition $\|H\varphi_1 - H\varphi_2\|_m \le L\|\varphi_1 - \varphi_2\|_n$ for $\varphi_1, \varphi_2 \in C^n\langle a,b\rangle$ must be affine, i.e., there are $G, H \in C^m\langle a,b\rangle$ such that $h(x,y) = G(x)y + H(x)$, $(x \in \langle a,b\rangle$, $y \in R)$.

Let $C^n\langle a,b\rangle$ be the Banach space of n-times continuously differentiable functions $\varphi: \langle a,b\rangle \to R$ with the norm

$$\|\varphi\|_n = \sum_{k=0}^{n-1} |\varphi^{(k)}(a)| + \sup_{\langle a,b\rangle} |\varphi^{(n)}|. \tag{1}$$

For the function

$$h: \langle a,b\rangle \times R \to R \tag{2}$$

define the so called operator of substitution or Nemytskii operator N generated by function (2), (cf. [2]),

$$(N\varphi)(x) = h(x, \varphi(x)); \quad (x \in \langle a,b\rangle,\ \varphi \in C^n\langle a,b\rangle). \tag{3}$$

The aim of this paper is to prove the following

Theorem.

Let m and n be positive integer and let $h: \langle a,b\rangle \times R \to R$ be of the class C^m in $\langle a,b\rangle \times R$. Then

1) in the case $m \le n$, the operator N maps $C^n\langle a,b\rangle$ into $C^m\langle a,b\rangle$ and the following two conditions are equivalent

(i) N is a Lipschitz map, i.e. there exists an $L > 0$ such that

$$\|N\varphi_1 - N\varphi_2\|_m \le L\,\|\varphi_1 - \varphi_2\|_n, \quad (\varphi_1, \varphi_2 \in C^n\langle a,b\rangle), \tag{4}$$

(ii) N is affine, i.e. there exist $G, H \in C^m\langle a,b\rangle$ such that

$$h(x,y) = G(x)y + H(x), \quad x \in \langle a,b\rangle,\ y \in R. \tag{5}$$

2) in the case $m > n$, N maps $C^n\langle a,b\rangle$ into $C^m\langle a,b\rangle$ iff N is constant, i.e. there is a $H \in C^m\langle a,b\rangle$ such that $h(x,y) = H(x)$ for $x \in \langle a,b\rangle$, $y \in R$.

Before the proof we introduce functions h_k inductively

$$h_0(x,y_0) = h(x,y_0),$$
$$h_{k+1}(x,y_0,\ldots,y_{k+1}) = \frac{\partial h_k}{\partial x}(x,y_0,\ldots,y_k) + \sum_{i=0}^{k} \frac{\partial h_k}{\partial y_i}(x,y_0,\ldots,y_k)y_{i+1}, \tag{6}$$

for $x \in \langle a,b\rangle$, $y_0, y_1, \ldots, y_{k+1} \in R$; $k=0,1,\ldots,m-1$.

We need the following two lemmas.

Lemma 1

If h is of class C^m in $\langle a,b\rangle \times R$ then h_k is of class C^{m-k} with respect to the variables $x \in \langle a,b\rangle$, $y_0 \in R$ and of class C^∞ with respect to $y_1, y_2, \ldots, y_k$, $(k=0,1,\ldots,m)$. Moreover, there exists a function $g_k: \langle a,b\rangle \times R^k \to R$ such that

$$h_k(x,y_0,\ldots,y_k) = g_k(x,y_0,\ldots,y_{k-1}) + \frac{\partial h}{\partial y}(x,y_0)y_k, \tag{7}$$

for $x \in \langle a,b\rangle$, $y_0, y_1, \ldots, y_k \in R$; $k=0,1,\ldots,m$.

Lemma 2

Suppose that h is of class C^m in $\langle a,b\rangle \times R$. If $\varphi \in C^m\langle a,b\rangle$ and $\Psi(x) = h(x, \varphi(x))$ for $x \in \langle a,b\rangle$ then

$$\psi^{(k)}(x) = h_k(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x)); \quad (x \in \langle a,b \rangle,\ k=0,\ldots,n).$$

Lemmas 1 and 2 follow immediately from (6) by induction (cf. [1]).

<u>Proof of the theorem</u>

1) Suppose that $m < n$. To prove the implication (i) ⇒ (ii) take $x, \bar{x} \in \langle a,b \rangle$, $x < \bar{x}$; $w_1, w_2 \in R$ and define the functions

$$\alpha_i(t) = \begin{cases} 0 & a \le t \le x \\ \dfrac{w_i}{(\bar{x}-x)^2}(t-x), & x \le t \le \bar{x}, \quad i=1,2. \\ \dfrac{w_i}{\bar{x}-x} & \bar{x} \le t \le b \end{cases} \tag{8}$$

It is easily seen that α_1, α_2 are continuous in $\langle a,b \rangle$ and

$$\sup_{\langle a,b \rangle} |\alpha_1 - \alpha_2| = \frac{|w_1 - w_2|}{\bar{x}-x}. \tag{9}$$

For arbitrary $z_1, z_2 \in R$ we define

$$\varphi_i(t) = z_i + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} \alpha_i(s)\,ds, \quad (t \in \langle a,b \rangle,\ i=1,2). \tag{10}$$

One can easily check that $\varphi_1, \varphi_2 \in C^n \langle a,b \rangle$ and

$$\varphi_i(t) = z_i; \quad \varphi_i^{(k)}(t) = 0, \quad (t \in \langle a,x \rangle,\ k=1,\ldots,n); \quad \varphi_i^{(n)} = \alpha_i, \quad (i=1,2). \tag{11}$$

Moreover, we have

$$\varphi_i^{(k)}(\bar{x}) = \frac{w_i}{(n-k+1)!}(\bar{x}-x)^{n-1-k}, \quad (k=1,\ldots,n;\ i=1,2), \tag{12}$$

which follows from (8) and (10) by a simple calculation.

Suppose now that (i) holds and put just constructed functions φ_1, φ_2 into the Lipschitz condition (4). By the definition of the norm in $C^n \langle a,b \rangle$ and $C^m \langle a,b \rangle$, by lemma 1 and 2 and by (11) inequality (4) may be written in the following form

$$\sum_{k=0}^{m-1} \left| h_k(a, z_1, 0, \ldots, 0) - h_k(a, z_2, 0, \ldots, 0) \right| +$$

$$+ \sup_{t \in \langle a,b \rangle} \left| g_m(t, \varphi_1(t), \ldots, \varphi_1^{(m-1)}(t)) + \frac{\partial h}{\partial y}(t, \varphi_1(t))\, \varphi_1^{(m)}(t) - \right.$$

$$- g_n(t, \varphi_2(t), \dots, \varphi_2^{(n-1)}(t)) - \frac{\partial h}{\partial y}(t, \varphi_2(t))\, \varphi_2^{(n)}(t) \Big| <$$

$$< L(|z_1 - z_2| + \sup_{\langle a,b \rangle} |\alpha_1 - \alpha_2|).$$

Omitting the component $\sum_{k=0}^{n-1}$ and the sign supremum on the left hand side of above inequality and next setting $t = \bar{x}$ we get by (12) and (9)

$$\Big| g_n(\bar{x}, \varphi_1(\bar{x}), \dots, \varphi_1^{(n-1)}(\bar{x})) - g_n(\bar{x}, \varphi_2(\bar{x}), \dots, \varphi_2^{(n-1)}(\bar{x})) +$$

$$+ c_n(\bar{x} - x)^{n-1-m} \Big[\frac{\partial h}{\partial y}(\bar{x}, \varphi_1(x)) w_1 - \frac{\partial h}{\partial y}(\bar{x}, \varphi_2(\bar{x})) w_2 \Big] \Big| <$$

$$< L \left(|z_1 - z_2| + \frac{|w_1 - w_2|}{\bar{x} - x} \right),$$

where w_1, w_2 are arbitrary real numbers and

$$c_k = \frac{1}{(n-k+1)!}; \quad (k = 1, \dots, n).$$

Multiplying both sides of this inequality by $\bar{x} - x$ and next putting

$$w_1 = w_2 = (\bar{x} - x)^{m-n}, \tag{13}$$

we get the following inequality

$$\Big| (\bar{x} - x) \Big[g_n(\bar{x}, \varphi_1(\bar{x}), \dots, \varphi_1^{(n-1)}(\bar{x})) - g_n(\bar{x}, \varphi_2(\bar{x}), \dots, \varphi_2^{(n-1)}(\bar{x})) \Big] +$$

$$+ c_n \Big[\frac{\partial h}{\partial y}(\bar{x}, \varphi_1(\bar{x})) - \frac{\partial h}{\partial y}(\bar{x}, \varphi_2(\bar{x})) \Big] \Big| < L(\bar{x} - x)\, |z_1 - z_2|.$$

It follows from (12) and (13) that

$$\varphi_i^{(k)}(\bar{x}) = c_k(\bar{x} - x)^{m-k-1}, \quad (k = 1, \dots, n-1).$$

Consequently, we have

$$\lim_{\bar{x} \to x} \varphi_i^{(k)}(\bar{x}) = \begin{cases} 0, & k = 1, \dots, m-2, \\ c_m, & k = m-1 \end{cases} \tag{14}$$

and by (10)

$$\lim_{\bar{x} \to x} \varphi_i(x) = z_i, \quad (i = 1, 2). \tag{15}$$

By (14) and (15), letting $\bar{x} \to x$ in the last inequality, we get

$$\frac{\partial h}{\partial y}(x, z_1) = \frac{\partial h}{\partial y}(x, z_2),$$

for all $x \in \langle a,b \rangle$ and $z_1, z_2 \in R$. This means that $\frac{\partial h}{\partial y}$ is constant with respect to the second variable and, consequently, there exist functions $G, H: \langle a,b \rangle \to R$ such that h has form (5). Since h is of class C^m in $\langle a,b \rangle \times R$, it follows that $G, H \in C^m\langle a,b \rangle$ which completes the proof of the implication (i) $\Rightarrow$ (ii).

To prove the converse implication take $\varphi_1, \varphi_2 \in C^m\langle a,b \rangle$ and put $\varphi = \varphi_1 - \varphi_2$. By (5) we have $N\varphi_1 - N\varphi_2 = G\cdot\varphi$. Since $G \in C^m\langle a,b \rangle$ and $m \le n$, it is easily seen that $\varphi \to G\cdot\varphi$ is a linear and continuous mapping of Banach space $C^m\langle a,b \rangle$ into $C^m\langle a,b \rangle$. Therefore there exists an $L>0$ such that $\|G\cdot\varphi\|_m \le L\,\|\varphi\|_m$ for all $\varphi \in C^m\langle a,b \rangle$. This completes the proof of 1).

2) For an indirect proof suppose that there is a point $(x_0, y_0) \in \langle a,b \rangle \times R$ such that $\frac{\partial h}{\partial y}(x_0, y_0) \ne 0$. Take a function $\varphi \in C^m\langle a,b \rangle$ such that $\varphi^{(n+1)}(x_0)$ does not exist and $\varphi(x_0) = y_0$. Since N maps $C^m\langle a,b \rangle$ into $C^m\langle a,b \rangle$, we have $\psi = N\varphi \in C^m\langle a,b \rangle$. Since $m > n$ it follows from lemmas 1 and 2 that

$$\varphi^{(n)}(x) = \frac{\psi^{(n)}(x) - g_n(x, \varphi(x), \dots, \varphi^{(n-1)}(x))}{\frac{\partial h}{\partial y}(x, \varphi(x))},$$

in some neighbourhood of x_0. Hence $\varphi^{(n)}$ is differentiable at the point x_0. This contradiction shows that $\frac{\partial h}{\partial y}(x,y) = 0$ for $x \in \langle a,b \rangle$, $y \in R$ which completes the proof.

Remark 1.

A similar theorem is true for the Banach space of Lipschitzian functions (cf. [3]). Note that no similar result holds for the Banach spaces $C\langle a,b \rangle$ and $L^p\langle a,b \rangle$. In fact, the function $h(x,y) = \sin y$ generates operators $N: C\langle a,b \rangle \to C\langle a,b \rangle$ and $N: L^p\langle a,b \rangle \to L^p\langle a,b \rangle$ such that in the both cases we have

$$\|N\varphi_1 - N\varphi_2\| \le \|\varphi_1 - \varphi_2\|.$$

Remark 2.

Our theorem explains why the Banach fixed point theorem cannot be directly applied in the theory of differentiable solutions of the nonlinear functional equation $\varphi(x) = h(x, \varphi[f(x)])$.

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POSTAĆ LIPSCHITZOWSKICH OPERATORÓW PODSTAWIENIA W PRZESTRZENI BANACHA FUNKCJI RÓŻNICZKOWALNYCH

Streszczenie

Niech $C^n\langle a,b\rangle$ będzie przestrzenią Banacha funkcji z ciągłą n-tą pochodną z naturalną normą, niech $h:\langle a,b\rangle \times R \to R$ będzie klasy C^m w $\langle a,b\rangle \times R$ i niech $N(\varphi)(x) = h(x, \varphi(x))$ dla $\varphi \in C^n\langle a,b\rangle$. W pracy tej dowodzi się, że jeżeli $N: C^n\langle a,b\rangle \to C^m\langle a,b\rangle$ oraz istnieje $L > 0$ takie, że $\|N(\varphi_1) - N(\varphi_2)\|_m \leq L\|\varphi_1 - \varphi_2\|_n$ dla $\varphi_1, \varphi_2 \in C^n\langle a,b\rangle$ to w przypadku 1) gdy $m \leq n$, funkcja h jest liniowa ze względu na drugą zmienną; 2) gdy $m > n$, h jest stała ze względu na drugą zmienną.

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ФОРМА ЛИПШИЦЕВСКИХ ОПЕРАТОРОВ ПОДСТАНОВКИ В ПРОСТРАНСТВЕ БАНАХА ДИФФЕРЕНЦИРУЕМЫХ ФУНКЦИЙ

Краткое содержание

Пусть $C^n\langle a, b\rangle$ будет пространством Банаха функций с непрерывной производной n с натуральной нормой; пусть $h: \langle a, b\rangle \times R \to R$ будет класса C^m в $\langle a, b\rangle \times R$ и пусть $N(\varphi)(x) = h(x, \varphi(x))$ для $\varphi \in C^n\langle a, b\rangle$. В работе доказывается, что если $N: C^n\langle a, b\rangle \to C^m\langle a, b\rangle$, а также существует $L > 0$ таков, что $\|N(\varphi_1) - N(\varphi_2)\|_m \leq L\|\varphi_1 - \varphi_2\|_n$ для $\varphi_1, \varphi_2 \in C^n\langle a, b\rangle$, то в случае 1) когда $m \leq n$ функция h является линейной при учёте второй переменной; 2) когда $m > n$, h является постоянной, принимая во внимание вторую переменную.

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