Funkcialaj Ekvacioj

(Serio Internacia)

VOLUMO 25, NUMERO 2
AŬGUSTO, 1982

Functional Equations and Nemytskii Operators

By

Janusz Matkowski

Coden FESIAT Pages 115–243
Volume 25 August 1982
Number 2 Whole No. 64

FAKO DE L'FUNKCIALAJ EKVACIOJ
JAPANA MATEMATIKA SOCIETO

Funkcialaj Ekvacioj, 25 (1982) 127-132

Functional Equations and Nemytskii Operators

By

Janusz Matkowski

(Polytechnical University Łódź, Poland)

1. In this paper we examine the so called Nemytskii operator of substitution which appears in a natural way when we are looking for a solution of the functional equation

$$\varphi(x)=h(x, \varphi[f(x)]) \tag{1}$$

where $f: [a, b]\to[a, b]$ and $h: [a, b]\times R\to R$ are given functions.

It is known (cf. [1]) that the existence as well as the quantity of solutions of equation (1) depends mainly on the class of regularity of the unknown function. Under some general assumptions, the basic theorems on the existence and uniqueness of the solution of equation (1) in the classes $L^p[a, b]$ and $C[a, b]$ can be proved by means of the classical Banach fixed point theorem (cf. [1], [3]). Contrary to these cases, the proof of the existence of the solution in the class Lip $[a, b]$ is a little more complicated and goes with using of the Schauder fixed point theorem. It is of interest that the uniqueness of such solution can be proved independently (cf. [4]). Hence, one can expect that in this case the proof should also go with using only the Banach fixed point theorem. It is important because of the possibility of determining of the solution as the limit with respect to the Lip $[a, b]$-norm of the sequence of the successive approximations.

We are going to present a theorem which shows that in the nonlinear case the Lipschitzian solution of equation (1) cannot be obtained by a direct application of Banach principle. We also prove that Banach's method can be applied for the linear equation

$$\varphi(x)=G(x)\varphi[f(x)]+H(x) \tag{2}$$

to find the Lipschitzian solution.

2. Suppose that we are looking for a solution $\varphi\in X$ of equation (1) where X is a Banach space of functions $\varphi: [a, b]\to R$ with the norm $\|\varphi\|$. This equation has the form

$$\varphi=(N\circ S)\varphi \tag{3}$$

where $S(\varphi)=\varphi\circ f$ is linear and, in general, continuous mapping from X into itself, and $N\colon X\to X$ given by the formula

$$(4)\qquad (N\varphi)(x)=h(x,\varphi(x)),\qquad \varphi\in X,$$

is the so called Nemytskii operator of substitution. Clearly, N is, in general, a non-linear operator.

To apply the Banach's principle to equation (3) it is necessary for $N\circ S$ to be a contraction map of X into itself. But this "practically" forces N to be a Lipschitz map, i.e. that there is an $L>0$ such that

$$(5)\qquad \|N(\varphi_1)-N(\varphi_2)\|\leq L\,\|\varphi_1-\varphi_2\|,\qquad \varphi_1,\varphi_2\in X.$$

It is easily seen that this is the case for every S which maps X onto X and is invertible. Indeed, by the open mapping theorem S^{-1} is continuous and we have

$$\|N\varphi_1-N\varphi_2\|\leq\|(N\circ S)(S^{-1}\varphi_1)-(N\circ S)(S^{-1}\varphi_2)\|\leq k\|S^{-1}\varphi_1-S^{-1}\varphi_2\|$$
$$=k\|S^{-1}(\varphi_1-\varphi_2)\|\leq k\|S^{-1}\|\,\|\varphi_1-\varphi_2\|,$$

for $\varphi_1,\varphi_2\in X$ and some $k<1$.

3. Now we confine our considerations to the case $X=\mathrm{Lip}\,[a,b]$ with the norm defined by the formula

$$(6)\qquad \|\varphi\|=|\varphi(a)|+\sup_{x\neq\bar{x}}\frac{|\varphi(x)-\varphi(\bar{x})|}{|x-\bar{x}|},\qquad \varphi\in\mathrm{Lip}\,[a,b],$$

where supremum is taken over all $x,\bar{x}\in[a,b]$. Of course $\mathrm{Lip}\,[a,b]$ is a Banach space.

We are going to prove the following

Theorem. *Suppose that $h\colon[a,b]\times R\to R$, $X=\mathrm{Lip}\,[a,b]$ and*

$$(7)\qquad N\colon \mathrm{Lip}\,[a,b]\longrightarrow\mathrm{Lip}\,[a,b]$$

where operator N is defined by (4). Under these conditions the operator N is Lipschitzian, i.e. N fulfils inequality (5), if and only if there exist functions $G,H\in$ $\mathrm{Lip}\,[a,b]$ such that

$$(8)\qquad h(x,y)=G(x)y+H(x),\quad x\in[a,b],\quad y\in R.$$

Proof. It follows from (7) that for every $y\in R$ the function $h(\cdot,y)\in\mathrm{Lip}\,[a,b]$. Therefore h is continuous with respect to the first variable. Suppose that N satisfies inequality (5). By (6) this inequality has the following form

$$(9)\qquad \begin{aligned} &|h(a,\varphi_1(a))-h(a,\varphi_2(a))| \\ &\quad+\sup_{t\neq \bar{t}}\frac{|h(t,\varphi_1(t))-h(t,\varphi_2(t))-h(\bar{t},\varphi_1(\bar{t}))+h(\bar{t},\varphi_2(\bar{t}))|}{|t-\bar{t}|} \\ &\leqslant L\left(|\varphi_1(a)-\varphi_2(a)|+\sup_{t\neq\bar{t}}\frac{|\varphi_1(t)-\varphi_2(t)-\varphi_1(\bar{t})+\varphi_2(\bar{t})|}{|t-\bar{t}|}\right)\end{aligned}$$

for all $\varphi_1, \varphi_2 \in \mathrm{Lip}\,[a, b]$.

Let us fix $x, \bar{x} \in [a, b]$; $x<\bar{x}$, and $y_1, y_2, \bar{y}_1, \bar{y}_2 \in R$. Define the functions

$$\varphi_i(t)=\begin{cases} y_i & a\leqslant t<x \\ \dfrac{y_i-\bar{y}_i}{x-\bar{x}}(t-\bar{x})+\bar{y}_i & x\leqslant t\leqslant \bar{x};\quad i=1,2 \\ \bar{y}_i & \bar{x}<t\leqslant b. \end{cases}$$

It is easily seen that $\varphi_i \in \mathrm{Lip}\,[a, b]$, $i=1, 2$ and

$$\|\varphi_1-\varphi_2\|=|y_1-y_2|+\frac{|y_1-y_2-\bar{y}_1+\bar{y}_2|}{|x-\bar{x}|}.$$

Omit the sign supremum on the left hand side of inequality (9). Putting in such obtained inequality φ_1, φ_2 defined above with $t=x$ and $\bar{t}=\bar{x}$ we get

$$|h(a,y_1)-h(a,y_2)|+\frac{|h(x,y_1)-h(x,y_2)-h(\bar{x},\bar{y}_1)+h(\bar{x},\bar{y}_2)|}{|x-\bar{x}|} \leqslant L\left(|y_1-y_2|+\frac{|y_1-y_2-\bar{y}_1+\bar{y}_2|}{|x-\bar{x}|}\right).$$

Multiplying both sides of this inequality by $|x-\bar{x}|$ and letting $\bar{x}\to x$ it follows from the continuity of $h(\cdot, y)$ that

$$(10)\qquad |h(x,y_1)-h(x,y_2)-h(x,\bar{y}_1)+h(x,\bar{y}_2)|\leqslant L\,|y_1-y_2-\bar{y}_1+\bar{y}_2|$$

for all $x \in [a, b]$ and $y_1, y_2, \bar{y}_1, \bar{y}_2 \in R$.

Let us fix an $x \in [a, b]$ and define the function $F_x\colon R\to R$ by the formula

$$(11)\qquad F_x(y)=h(x,y)-h(x,0),\qquad y\in R.$$

Setting $y_1=w+z$, $y_2=w$, $\bar{y}_1=z$, $\bar{y}_2=0$, $w, z \in R$, in (10) we get

$$h(x,w+z)=h(x,w)+h(x,z)-h(x,0),$$

which using (11) can be written in the following form

$$F_x(w+z)=F_x(w)+F_x(z),\qquad w, z\in R.$$

Thus F_x is additive. Setting $\bar{y}_1=\bar{y}_2=0$ in (10) we get by (11)

$$|F_x(y_1)-F_x(y_2)|\leqslant L\,|y_1-y_2|, \qquad y_1, y_2\in R$$

and consequently F_x is continuous. Since every additive and continuous function is linear there exists a function $G\colon [a,b]\to R$ such that

$$F_x(y)=G(x)y, \qquad y\in R.$$

Setting

$$H(x)=h(x,0), \qquad x\in[a,b]$$

it follows from (11) that

$$h(x,y)=G(x)y+H(x), \quad x\in[a,b], \quad y\in R,$$

and evidently we have $H\in \mathrm{Lip}\,[a,b]$. Since

$$G(x)=h(x,1)-h(x,0), \qquad x\in[a,b],$$

we also have $G\in \mathrm{Lip}\,[a,b]$.

Now let $G, H\in \mathrm{Lip}\,[a,b]$ and suppose that Nemytskii operator N is generated by function (8). One can easily verify that

(12) $$\|N\varphi_1-N\varphi_2\|\leqslant(\alpha+\gamma\beta)\,\|\varphi_1-\varphi_2\|, \qquad \varphi_1,\varphi_2\in \mathrm{Lip}\,[a,b]$$

where $\alpha=\sup_{[a,b]}|G|$, $\beta=\max(1, b-a)$ and

$$\gamma=\sup_{x\neq\bar{x}}\frac{|G(x)-G(\bar{x})|}{|x-\bar{x}|}.$$

Thus N is a Lipschitz map and this completes the proof.

4. Remarks to the Theorem.

(i) From the Theorem it follows that every Lipschitz and Nemytskii operator $N\colon \mathrm{Lip}\,[a,b]\to \mathrm{Lip}\,[a,b]$ such that $N(0)=0$ must be a linear map.

(ii) Note that condition $N\colon \mathrm{Lip}\,[a,b]\to \mathrm{Lip}\,[a,b]$ follows from (5) assuming that $N(0)\in \mathrm{Lip}\,[a,b]$. In fact, we have for $\varphi\in \mathrm{Lip}\,[a,b]$

$$\|N(\varphi)\|\leqslant\|N(\varphi)-N(0)\|+\|N(0)\|\leqslant L\,\|\varphi\|+\|N(0)\|<\infty.$$

(iii) It is easily seen that the Theorem remains valid if we change $[a,b]$ by the open interval (a,b), $-\infty<a<b<\infty$. However it is not longer true for the case $a=-\infty$ or $b=\infty$. To see this consider the Banach space $\mathrm{Lip}\,(R)$ with the norm

$$\|\varphi\|=|\varphi(0)|+\sup_{x\neq\bar{x}}\frac{|\varphi(x)-\varphi(\bar{x})|}{|x-\bar{x}|}, \qquad (x,\bar{x}\in R).$$

By the same reasoning as in the proof of Theorem we get that every Nemytskii operator N: Lip $(R)\to$Lip (R) have to be generated by the function $h(x, y)=G(x)y+H(x)$ where $G, H \in$ Lip (R). But the converse is not true. In fact, taking $h(x, y)=(\sin x)y$ and $\varphi(x)=x$ for $x, y \in R$, we have $G(x)=\sin x$, $H=0$; $G, H, \varphi \in$ Lip (R) and $N(0)=0$. Since $N(\varphi)=G\varphi$ $(x\to x \sin x)$ does not belong to Lip (R) it follows from the remark (ii) that N is not a Lipschitz map.

But we can give the following sufficient condition for N: Lip $(R)\to$Lip (R) to be a Lipschitz map. Let $G, H \in$ Lip (R) and suppose that

$$\alpha=\sup_{x\neq\bar{x}} \frac{|G(x)-G(\bar{x})|}{|x-\bar{x}|}(1+|x|)<\infty, \qquad \beta=\sup_{x\in R} |G(x)|<\infty.$$

Then operator N generated by the function $h(x, y)=G(x)y+H(x)$ satisfies the following Lipschitz condition

$$\|N\varphi_1-N\varphi_2\|\leqslant(\alpha+\beta)\|\varphi_1-\varphi_2\|, \qquad \varphi_1, \varphi_2 \in \text{Lip}\,(R).$$

(iv) Note that in the Theorem the space Lip $[a, b]$ cannot be replaced by $C[a, b]$ or $L^p[a, b]$. In fact, we have for the Nemytskii operator N generated by the function $h(x, y)=\sin y$

$$\|N\varphi_1-N\varphi_2\|=\sup_{[a,b]}|\sin\varphi_1(x)-\sin\varphi_2(x)|\leqslant\|\varphi_1-\varphi_2\|, \qquad \varphi_1, \varphi_2 \in C[a, b],$$

and also

$$\|N\varphi_1-N\varphi_2\|^p=\int_a^b|\sin\varphi_1(t)-\sin\varphi_2(t)|^p\,dt\leqslant\int_a^b|\varphi_1(t)-\varphi_2(t)|^p\,dt=\|\varphi_1-\varphi_2\|^p$$

for $\varphi_1, \varphi_2 \in L^p[a, b]$. Thus in the both cases N is a nonlinear Lipschitz map.

(v) Taking into account considerations of section 2 and inequality (12) one can easily establish the conditions under which the Banach fixed point is applicable to equation (2).

(vi) The continuity and boundedness properties of nonlinear Nemytskii operators acting in spaces of integrable functions are considered in Martin's book [2].

References

[1] Kuczma, M., Functional equations in a single variable, PWN, Warszawa, (1968).

[2] Martin, R. H., Jr., *Nonlinear operators and differential equations in Banach spaces*, New York, 1976.

[3] Matkowski, J., Integrable solutions of functional equations, Dissertationes Math., **CXXVII** (1975), 1–68.

[4] ——, On Lipschitzian solutions of functional equation, Ann. Polon. Math., **28** (1973), 135–139.

nuna adreso:
Lubertowicza 3/1
43-300 Bielsko-Biała
Poland

(Ricevita la 10-an de junio, 1981)