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FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS IN METRIC SPACES

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1. In this paper we extend Banach's and Kannan's fixed point theorems as well as some results of D. W. BOYD and J. S. W. WONG, A. MEIR and E. KEELER, S. REICH and C. S. WONG.

Our main result is the following

Theorem 1. *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that for every $\varepsilon > 0$ and $x, y \in X$,*

(1) $$0 < d(Tx, x), d(y, Ty), d(x, y), \frac{d(Tx, y) + d(x, Ty)}{2} \leqq \varepsilon \Rightarrow d(Tx, Ty) < \varepsilon .$$

If for every $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$,

(2) $$\left.\begin{array}{c} \varepsilon < d(x, y) < \varepsilon + \delta \\ 0 < d(Tx, x), \dfrac{d(Tx, y) + d(x, Ty)}{2} \text{[illegible]}, d(y, Ty) < \varepsilon + \delta \end{array}\right\} \Rightarrow d(Tx, Ty) \leqq \varepsilon ,$$

then for every $x \in X$, the sequence $\{T^n x\}$ converges. If, moreover, T is continuous or, given $\varepsilon > 0$, there is a μ, $0 < \mu < \varepsilon$ such that for $x, y \in X$

(3) $$\left.\begin{array}{c} 0 < d(Tx, x), \dfrac{d(Tx, y) + d(x, Ty)}{2} \leqq \varepsilon \\ 0 < d(x, y), \quad d(y, Ty) < \mu \end{array}\right\} \Rightarrow d(Tx, Ty) < \varepsilon - \mu ,$$

then T has a unique fixed point $p \in X$ and for every $x \in X$, $\lim_{n \to \infty} T^n x = p$.

Proof. Take an $x \in X$ and put $x_n = T^n x$, $n = 0, 1, \ldots$. We can assume that $d(x_n, x_{n-1}) > 0$, $n = 1, 2, \ldots$. Note that

(4) $$d(x_{n+1}, x_n) < d(x_n, x_{n-1}), \quad n = 1, 2, \ldots$$

For an indirect proof of (4) suppose that $d(x_{n+1}, x_n) \geqq d(x_n, x_{n-1})$ for some $n \geqq 1$. Assuming $x = x_n$, $y = x_{n-1}$, $\varepsilon = d(x_{n+1}, x_n)$ we have $d(Tx, x) = \varepsilon$, $d(x, y) = d(y, Ty) \leqq \varepsilon$ and

$$d(Tx, y) + d(x, Ty) = d(x_{n+1}, x_{n-1}) \leqq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) \leqq 2\varepsilon .$$

Hence, using (1), we get $d(Tx, Ty) = d(x_{n+1}, x_n) < \varepsilon$. This contradiction proves (4) and, consequently, the sequence $\{d(x_{n+1}, x_n)\}$ converges. We shall show that

$$c = \lim_{n \to \infty} d(x_{n+1}, x_n) = 0 . \qquad (5)$$

Suppose that $c > 0$. Then there is n_0 such that

$$c < d(x_{n+1}, x_n) < c + \delta(c) , \quad n \geqq n_0 .$$

Using (2) for $x = x_{n+1}$, $y = x_n$ we hence obtain $d(x_{n+2}, x_{n+1}) \leqq c$, $n \geqq n_0$. This contradicts the previous inequality and proves (5).

Let us fix an $\varepsilon > 0$. Without loss of generality we can assume

$$\delta = \delta(\varepsilon) < \varepsilon . \qquad (6)$$

By (5) there is a k such that

$$d(x_{n+1}, x_n) < \tfrac{1}{2}\delta , \quad n \geqq k . \qquad (7)$$

We shall prove that

$$d(x_{n+m}, x_n) < \varepsilon + \tfrac{1}{2}\delta , \quad n \geqq k , \qquad (8)$$

for $m = 1, 2, \ldots$. By (7), this is the case for $m = 1$. Suppose that the inequalities (8) hold for some $m \geqq 1$. If $d(x_{n+m}, x_n) \leqq \varepsilon$, then by (7)

$$d(x_{n+m+1}, x_n) \leqq d(x_{n+m+1}, x_{n+m}) + d(x_{n+m}, x_n) < \varepsilon + \tfrac{1}{2}\delta .$$

If $\varepsilon < d(x_{n+m}, x_n) < \varepsilon + \frac{1}{2}\delta$ then, by (7), we have for $x = x_{n+m}$, $y = x_n$

$$\varepsilon < d(x, y) < \varepsilon + \tfrac{1}{2}\delta , \quad d(Tx, x) < \tfrac{1}{2}\delta , \quad d(y, Ty) < \tfrac{1}{2}\delta ,$$

$$0 < d(Tx, y) + d(x, Ty) \leqq d(Tx, x) + 2d(x, y) + d(y, Ty) < 2(\varepsilon + \delta) .$$

Now (2) yields $d(Tx, Ty) = d(x_{n+m+1}, x_{n+1}) \leqq \varepsilon$. Thus

$$d(x_{n+m+1}, x_n) \leqq d(x_{n+m+1}, x_{n+1}) + d(x_{n+1}, x_n) < \varepsilon + \tfrac{1}{2}\delta ,$$

and induction completes the proof of (8).

Now (8) and (6) imply that $\{x_n\}$ is a Cauchy sequence and, since X is complete, $\{x_n\}$ converges to a point $p \in X$.

Suppose that the condition (3) holds and $\varepsilon = d(Tp, p) > 0$. By the preceding part of the proof we can find n_0 such that $d(p, x_n) < \frac{1}{2}\mu$, $d(x_{n+1}, x_n) < \frac{1}{2}\mu$ for $n \geqq n_0$. Hence, assuming $x = p$, $y = x_n$, we have $d(Tx, x) = \varepsilon$, $d(x, y) < \mu$, $d(y, Ty) < \mu$ and

$$d(Tx, y) + d(x, Ty) \leqq d(Tp, p) + d(p, x_n) + d(p, x_{n+1}) \leqq \varepsilon + \mu < 2\varepsilon .$$

Using (3) we obtain $d(Tx, Ty) = d(Tp, Tx_n) < \varepsilon - \mu$. This implies

$$d(Tp, p) \leqq d(Tp, Tx_n) + d(x_{n+1}, p) < \varepsilon - \mu + \tfrac{1}{2}\mu < \varepsilon,$$

which is a contradiction and therefore $Tp = p$.

The uniqueness of the fixed point follows from (1).

Remark 1. Suppose that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\varepsilon \leqq d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) < \varepsilon$, $x, y \in X$. Then (1) and (2) are fulfilled and T is continuous. Thus Theorem 1 generalizes the result of Meir and Keeler [2].

2. We apply theorem 1 to obtain a fixed point theorem which generalizes some results of Boyd and Wong [1], Reich [3] and Wong [4].

Theorem 2. *Let (X, d) be a complete metric space and let $T: X \to X$. Suppose that there exists a function $\alpha : \langle 0, \infty)^4 \to \langle 0, \infty)$ such that*

(9) $$d(Tx, Ty) \leqq \alpha\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(x, Ty)}{2}\right), \quad x, y \in X;$$

(10) $\alpha(t, s_1, s_2, s_3)$ *is increasing with respect to* s_1, s_2, s_3;

(11) $$0 < t \leqq s \Rightarrow \alpha(t, s, s, s) < s;$$

(12) $$\limsup_{t,u \to s+} \alpha(t, u, u, u) < s \quad \text{for} \quad s > 0.$$

Then for every $x \in X$, the sequence $\{T^n x\}$ converges. If, moreover,

(13) $$\limsup_{t,u \to 0+} \alpha(t, s, u, s) < s \quad \text{for} \quad s > 0,$$

then T has a unique fixed point $p \in X$ and $\lim T^n x = p$ for $x \in X$.

Proof. Take an $\varepsilon > 0$ and note that

(14) $$0 < t_i \leqq \varepsilon \ (i = 1, 2, 3, 4) \Rightarrow \alpha(t_1, t_2, t_3, t_4) < \varepsilon.$$

In fact, if $t_1 \geqq \max(t_2, t_3, t_4)$ then by (10)–(11) we have

$$\alpha(t_1, t_2, t_3, t_4) \leqq \alpha(t_1, t_1, t_1, t_1) < t_1 \leqq \varepsilon.$$

If $t_1 < s = \max(t_2, t_3, t_4)$ then by (10)–(11) we have

$$\alpha(t_1, t_2, t_3, t_4) \leqq \alpha(t_1, s, s, s) < s \leqq \varepsilon$$

which proves (14).

It follows from (12) that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\varepsilon < u, \quad s < \varepsilon + \delta \Rightarrow \alpha(u, s, s, s) < \varepsilon.$$

Hence and from (10)–(11) we easily obtain

(15) $$\varepsilon < t_1 < \varepsilon + \delta, \quad 0 < t_2, t_3, t_4 < \varepsilon + \delta \Rightarrow \alpha(t_1, t_2, t_3, t_4) < \varepsilon.$$

Finally, the condition (13) implies that for every $\varepsilon > 0$ there is a μ, $0 < \mu < \varepsilon$, such that

$$(16) \qquad 0 < t_1, t_3 < \mu \Rightarrow \alpha(t_1, \varepsilon, t_3, \varepsilon) < \varepsilon - \mu .$$

To prove (16) we put $s = \limsup_{t_1, t_3 \to 0} \alpha(t_1, \varepsilon, t_3, \varepsilon)$. By (13) there is a $\bar{\mu} > 0$ such that for $0 < t_1, t_3 < \bar{\mu}$ we have $\alpha(t_1, \varepsilon, t_3, \varepsilon) < \varepsilon - \frac{1}{2}(\varepsilon - s)$. Evidently, $\mu = \min(\bar{\mu}, \frac{1}{2}(\varepsilon - s))$ satisfies the condition (16).

Now, setting $t_1 = d(x, y)$, $t_2 = d(x, Tx)$, $t_3 = d(y, Ty)$, $t_4 = \frac{1}{2}(d(x, Ty) + d(Tx, y))$ and taking into account (9) and (14)–(16) we see that all the assumptions of Theorem 1 are fulfilled. This completes the proof.

Remark 2. If α does not depend on t_2, t_3, t_4 then the condition (9) takes the form $d(Tx, Ty) \leq \gamma(d(x, y))$, $x, y \in X$. Suppose that $\gamma(t) < t$ for $t > 0$ and γ is upper semicontinuous from the right. Then the conditions (10)–(13) are fulfilled, and, consequently, Theorem 2 implies the result of Boyd and Wong [1].

Theorem 2 generalizes also the results of S. Reich ([3], Th. 1) and C. S. Wong ([4], Th. 1).

Remark 3. In this paper "increasing" means nondecreasing. Note that in Theorem 2 the function α need not be increasing with respect to the first variable (cf. Remark 2).

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