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INTEGRABLE SOLUTIONS OF A LINEAR FUNCTIONAL INEQUALITY

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ABSTRACT. In this note we establish conditions on functions f and g under which the functional inequality $|\varphi(x)| \leq g(x)|\varphi(f(x))|$ has no nontrivial integrable solutions.

1. INTRODUCTION

We shall consider integrable solutions of the linear functional inequality in a single variable

(1) $$|\varphi| \leq g|\varphi \circ f|,$$

where f and g are given, φ is an unknown function, and $g|\varphi \circ f|$ denotes the function $x \to g(x)|\varphi(f(x))|$. The inequality (1) appears in a natural way in the problem of the uniqueness of a general functional equation of the first order (cf. [2]).

Integrable solutions of the corresponding functional equation

(2) $$\varphi = g\,\varphi \circ f$$

were considered by M. Kuczma [1] and in [2].

2. CONDITIONS

We assume the following:

(*) (X, S, μ) is a σ-finite measure space, $\mu(X) > 0$; $f: X \to X$ is one-to-one; f and f^{-1} are S-measurable; the measures μf^{-1} and μf are absolutely continuous with respect to μ; and

(3) $$\mu\Big(X \setminus \bigcup_{k=0}^{\infty} [f^k(X) \setminus f^{k+1}(X)]\Big) = 0.$$

REMARK. (1) Let us note that if $X = (0,1)$, μ is Lebesgue measure, and $f: (0,1) \to (0,1)$ satisfies the condition

$$0 < f(x) < x, \quad x \in (0,1),$$

then (3) is fulfilled. In particular, it follows from (3) that the set of fixed points of f has measure zero.

(ii) Conditions (*) imply that every two measures in the sequence μf^n, $n = 0, \pm 1, \dots$, are absolutely continuous with respect to each other. Moreover, if $\mu(X) < \infty$, then $\lim_{n\to\infty} \mu f^n(X) = 0$.

In fact, if $\mu f^n(A) = 0$, then by (*) we have

$$\mu f^{n-1}(A) = \mu f^{-1}(f^n(A)) = 0, \quad \mu f^{n+1}(A) = \mu f(f^n(A)) = 0,$$

which implies the first statement contained in Remark (ii). To prove the second one, note that $f^{n+1}(X) \subset f^n(X)$, $n = 0,1,\dots$. Therefore

$$\bigcap_{n=0}^{\infty} f^n(X) \subset \{X \setminus \bigcup_{n=0}^{\infty} (f^n(X)\setminus f^{n+1}(X))\},$$

and, consequently, by (3), we have

$$\lim_{n\to\infty} \mu f^n(X) = \mu\left(\bigcap_{n=0}^{\infty} f^n(X)\right) = 0.$$

3. SOLUTION

Put

$$A_n = f^n(X)\setminus f^{n+1}(X), \quad n = 0,1,\dots . \tag{4}$$

For the function $g: X \to \mathbb{R}$, we write

$$Y_g = \{x \in X : g(x) = 0\}$$

and

$$\kappa_n = \operatorname*{ess\,inf}_{A_n \setminus Y_g} g^{-p} \frac{d\mu f}{d\mu}, \quad n = 0,1,\dots . \tag{5}$$

THEOREM. Let (*) be fulfilled and let $g: X \to \mathbb{R}$ be measurable and nonnegative. If the series

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \kappa_i \tag{6}$$

diverges and $\varphi \in L^p(X,S,\mu)$ satisfies inequality (1) a.e. in X, then $\varphi = 0$ a.e. in X.

Proof. By (*) and (4), we have

$$A_n \in S, \quad f(A_n) = A_{n+1}, \quad f^{-1}(A_{n+1}) = A_n, \quad A_n \cap A_k = \emptyset,$$

$n \neq k$; $n,k = 0,1,\dots$. Moreover, it follows from (3) that

$$\int_X |\varphi|^p d\mu = \sum_{n=0}^{\infty} \int_{A_n} |\varphi|^p d\mu, \quad \varphi \in L^p(X,S,\mu). \tag{7}$$

Suppose that $\varphi \in L^p(X,S,\mu)$ satisfies (1) a.e. in X. Note that by (1) we have $\varphi = 0$ a.e. in N_g. From (*), (1), and (5), we obtain

$$\int_{A_{n+1}} |\varphi|^p d\mu = \int_{f^{-1}(A_{n+1})} |\varphi\circ f|^p d\mu f = \int_{A_n} |\varphi\circ f|^p \frac{d\mu f}{d\mu}\, d\mu$$

$$\geq \int_{A_n \setminus N_g} \frac{d\mu f}{d\mu}\, g^{-p}\, |\varphi|^p d\mu \geq \varepsilon_n \int_{A_n} |\varphi|^p d\mu .$$

Having repeated this procedure n times, we arrive at the inequality

$$\int_{A_{n+1}} |\varphi|^p d\mu \geq \Big(\prod_{i=0}^{n} \varepsilon_i\Big) \int_{A_0} |\varphi|^p d\mu .$$

Hence, using (7), we have

$$\infty > \int_X |\varphi|^p d\mu \geq \Big(\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varepsilon_i\Big) \int_{A_0} |\varphi|^p d\mu .$$

Now the divergence of the series (6) implies $\varphi = 0$ a.e. in A_0. Since (cf. [1])

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varepsilon_i = \sum_{n=0}^{k} \prod_{i=0}^{n-1} \varepsilon_i + \Big(\prod_{i=0}^{k} \varepsilon_i\Big) \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varepsilon_{k+i+1},$$

the series $\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varepsilon_{k+i+1}$ diverges for $k = 1,2,\ldots$. Replacing A_0 by A_k and X by $f^k(X)$ in the above argumentation, we obtain $\varphi = 0$ a.e. in A_k for $k = 1,2,\ldots$. In view of (3) and (4), we have $\mu(X \setminus \bigcup_{n=0}^{\infty} A_n) = 0$ and, consequently, $\varphi = 0$ a.e. in X. This completes the proof.

REFERENCES

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