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ON SEQUENCES OF CONTRACTION MAPPINGS

BY

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In [4] the second author of this paper shows that if T is a self mapping of a complete metric space $(\mathcal{X}, d)$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in \mathcal{X}$

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(T(x), T(y)) \leqslant \varepsilon,$$

and

$$d(T(x), T(y)) < d(x, y) \text{ whenever } x \neq y,$$

then T has a unique fixpoint ξ and for every $x \in \mathcal{X}$

$$\lim_{n\to\infty} T^n(x) = \xi$$

(cf. [5]). Here we are going to prove the theorem on the convergence of fix points obtained from this theorem for a sequence of transformations. Namely, we have (cf. [6] and [7]).

THEOREM 1. *Let $(\mathcal{X}, d)$ be a complete metric space and T_n, $n = 0, 1, 2, \ldots$, yield self mappings of it such that for every $x \in \mathcal{X}$*

$$\lim_{n\to\infty} T_n(x) = T_0(x).$$

If given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in \mathcal{X}$ and $n = 0, 1, 2, \ldots$

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(T_n(x), T_n(y)) \leqslant \varepsilon$$

and

$$d(T_n(x), T_n(y)) < d(x, y) \text{ whenever } x \neq y;$$

then T_n has a unique fixpoint ξ_n,

$$\lim_{m\to\infty} T_n^m(x) = \xi_n$$

for every $x \in \mathcal{X}$, $n = 0, 1, 2, \ldots$, and

$$\lim_{n\to\infty} \xi_n = \xi_0.$$

Proof (cf. the proof of Theorem 2 in [4]). Let ξ_n be the unique fix point of T_n, $n = 1, 0, 2, \ldots$, and let us take an arbitrary positive real ε and corresponding δ for it. Without loss of generality we may assume that $\delta < \varepsilon$. Finally, for this δ find a positive integer N such that $d(T_n(\xi_0), \xi_0) < \delta$ for every integer $n \geq N$. Now, for every $x \in \mathfrak{X}$, if $d(x, \xi_0) < \varepsilon + \delta$ then $d(T_n(x), \xi_0) \leq d(T_n(x), T_n(\xi_0)) + d(T_n(\xi_0), \xi_0) < \varepsilon + \delta$ for every integer $n \geq N$ independent of whether the condition $d(x, \xi_0) \leq \varepsilon$ occurs or not. This shows that for every integer $n \geq N$ the map T_n transforms the ball $\{x \in \mathfrak{X} : d(x, \xi_0) < \varepsilon + \delta\}$ into itself. Hence $d(\xi_n, \xi_0) \leq \varepsilon + \delta$ for every integer $n \geq N$ and our theorem is proved.

The result just presented may be applied to the theory of functional equations of a single variable. Suppose to this end that

(i) (X, ϱ) and (Y, σ) are metric and complete metric spaces, respectively;

(ii) $h_n : X \times Y^S \to Y$ *) and $f_{s,n} : X \to X$, $n = 0, 1, 2, \ldots, s \in S$, where S is an arbitrary set, are continuous functions;

(iii) the sequence $\{h_n : n = 1, 2, \ldots\}$ tends uniformly on every compact set to h_0; for every compact subset C of X and positive real ε, there exists a positive integer N such that for every integer $n \geq N$ and an arbitrary $(s, x) \in S \times C$

$$\varrho(f_{s,n}(x), f_{s,0}(x)) < \varepsilon;$$

(iv) given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $(x, y), (x, \bar{y}) \in X \times Y^S$ and $n = 0, 1, 2, \ldots$

$\sigma(y(s), \bar{y}(s)) < \varepsilon + \delta$ for every $s \in S$ implies $\sigma(h_n(x, y), h_n(x, \bar{y})) \leq \varepsilon$. Moreover, for every two different (x, y) and $(x, \bar{y})$ from $X \times Y^S$ and for $n = 0, 1, 2, \ldots$ there exists an $s \in S$ such that

$$\sigma(h_n(x, y), h_n(x, \bar{y})) < \sigma(y(s), \bar{y}(s));$$

(v) there exists a sequence $\{G_m : m = 0, 1, 2, \ldots\}$ of open sets such that $X = \bigcup \{G_m : m = 0, 1, 2, \ldots\}$, $G_m \subset G_{m+1}$ and $\bar{G}_m$ is compact, $m = 0, 1, 2, \ldots$ Furthermore, $f_{s,n}(G_m) \subset G_m$ for every $s \in S$ and $n, m = 0, 1, 2, \ldots$.

THEOREM 2. *Under the hypotheses* (i)–(v) *equation*

$$\varphi(x) = h_n(x, \triangle_{s \in S} \varphi \circ f_{s,n}(x))\ ^{**)},$$

has exactly one continuous solution $\varphi_n : X \to Y$ *given by the formula*

$$\varphi_n = \lim_{m \to \infty} \varphi_{n,m},$$

where

$$\varphi_{n,m+1}(x) = h_n(x, \triangle_{s \in S} \varphi_{n,m} \circ f_{s,n}(x)), \ x \in X, \ m = 0, 1, 2, \ldots$$

*) By Y^S we denote the set of all functions from S into Y (with the Tychonoff topology).
**) By $\triangle_{s \in S} g_s$ we denote the diagonal of a family of transformations $\{g_s : s \in S\}$.

and $\varphi_{\nu,0}$ is an arbitrary continuous map from X into Y, $\nu = 0, 1, 2, \ldots$ ... Moreover, the sequence $\{\varphi_\nu : \nu = 1, 2, \ldots\}$ tends to φ_0 uniformly on every compact set.

The proof of this theorem does not differ essentially from that given in [1]–[3].

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