

# INVARIANCE OF THE BAJRAKTAREVIĆ MEANS WITH RESPECT TO THE BECKENBACH-GINI MEANS

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ABSTRACT. Assuming that the involved functions are three times differentiable, we determine all Bajraktarević means which are invariant respect to the mean-type mapping of the Beckenbach-Gini mean type. Some applications in iteration theory and functional equation are presented.

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## 1. Introduction

A function  $M: I^2 \rightarrow \mathbb{R}$  is called a *mean* in an interval  $I \subseteq \mathbb{R}$ , if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

The mean  $M$  is called *strict*, if these inequalities are sharp for all  $x, y \in I$ ,  $x \neq y$ ; and  $M$  is called *symmetric*, if  $M(x, y) = M(y, x)$  for all  $x, y \in I$ . If  $M$  is a mean in  $I$  then  $M(J^2) = J$  for every subinterval  $J \subseteq I$ , and, obviously,  $M$  is *reflexive*, i.e.

$$M(x, x) = x, \quad x \in I.$$

Let  $M, N: I^2 \rightarrow I$  be means. A mean  $K: I^2 \rightarrow I$  is called *invariant with respect to the mean-type mapping  $(M, N): I^2 \rightarrow I^2$*  (briefly,  $K$  is  $(M, N)$ -invariant, if

$$K(M(x, y), N(x, y)) = K(x, y), \quad x, y \in I.$$

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The mean  $K$  is also referred to as the Gauss composition of the means  $M$  and  $N$ . The invariant mean is useful when we are looking for the limits of the sequence of iterates of the mean-type mapping  $(M, N): I^2 \rightarrow I^2$  (cf. Borwein–Borwein [3], Bullen–Mitrinović–Vasić [4], also [13], [14], [15]). For example, note that

$$G(A(x, y), H(x, y)) = G(x, y), \quad x, y > 0,$$

where  $A, G, H$  denote, respectively, the arithmetic, geometric and harmonic mean, that is  $G$  is invariant with respect to the mean-type mapping  $(A, H)$ . This allows to obtain the following nontrivial fact:

$$\lim_{n \rightarrow \infty} (A, H)^n = (G, G),$$

where  $(A, H)^n$  denotes the  $n$ th iterate of the mapping  $(A, H)$ . Note also that the  $(A, H)$ -invariance of  $G$  is equivalent to the classical Pythagorean harmony proportion.

Let  $\mathcal{M}, \mathcal{N}, \mathcal{K}$  be three classes of means in the interval  $I$ . Consider the following problem. Determine all means  $M \in \mathcal{M}, N \in \mathcal{N}$  and  $K \in \mathcal{K}$  such that  $K$  is  $(M, N)$ -invariant. In the case when  $\mathcal{M} = \mathcal{N} = \mathcal{K} = \mathcal{A}$  where  $\mathcal{A}$  is the class of weighted quasi-arithmetic means this problem leads to the functional equation

$$A_p^{[f]}(x, y) + A_{1-p}^{[g]}(x, y) = x + y, \quad x, y \in I,$$

where  $f, g: I \rightarrow \mathbb{R}$  are the continuous and strictly monotonic functions,  $p \in (0, 1)$  is a fixed number and  $A_p^{[f]}: I^2 \rightarrow I$  defined by

$$A_p^{[f]}(x, y) := f^{-1}(pf(x) + (1-p)f(y)), \quad x, y \in I,$$

is a *weighted quasi-arithmetic mean* of a generator  $f$  and the weight  $p$ . In the case when  $p = \frac{1}{2}$ , the analytic solutions  $f, g$  of this equation were examined by O. Sutô [15], twice continuously differentiable solutions by J. Matkowski [14], and the continuously differentiable solutions by Z. Daróczy and Gy. Maksa [5]. A complete solution in the case  $p = \frac{1}{2}$  was done by Z. Daróczy and Zs. Páles [7]. The invariance problem in the case when  $p \in (0, 1)$  is arbitrary was first treated in J. Jarczyk and J. Matkowski [11] in the class of twice continuously differentiable functions and, recently, has been completely solved by J. Jarczyk [9].

Let the functions  $f, g: I \rightarrow \mathbb{R}$  be continuous,  $g(x) \neq 0$  for  $x \in I$ , and such that  $\frac{f}{g}$  is one-to-one. Then the function  $B^{[f,g]}: I^2 \rightarrow I$  defined by

$$B^{[f,g]}(x, y) = \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x) + f(y)}{g(x) + g(y)}\right), \quad x, y \in I,$$

is a mean in  $I$  and it is called Bajraktarević mean of generators  $f$  and  $g$  (cf. [2]).  $B^{[f,g]}$  is a strict mean, and it is a generalization of quasi-arithmetic mean. Note that in the case when  $\frac{f}{g} = \text{id}|_I$  we have  $B^{[f,g]} = B^{[g]}$  where

$$B^{[g]}(x, y) := \frac{xg(x) + yg(y)}{g(x) + g(y)}, \quad x, y \in I.$$

The mean  $B^{[g]}$  is called *Beckenbach-Gini mean* of a generator  $g$  (cf. [4]).

In the present paper we deal with the invariance problem  $K \circ (M, N) = K$  in the case when  $K = B^{[f,g]}$ ,  $M = B^{[f]}$  and  $N = B^{[g]}$ , i.e. we consider the composite functional equation

$$B^{[f,g]} \circ (B^{[f]}, B^{[g]}) = B^{[f,g]}.$$

Solving some differential equations, we prove that three times differentiable functions  $f$  and  $g$  satisfy this equation if, and only if, there is  $p \in \mathbb{R}$ ,  $p \neq 0$ , such that

$$f(x) = ae^{px}, \quad g(x) = be^{-px}, \quad x \in I.$$

Applying this result we show that, for a fixed  $p \in \mathbb{R}$ ,  $p \neq 0$ , a function  $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ , continuous on the diagonal  $\{(x, x) : x \in \mathbb{R}\}$ , satisfies the functional equation

$$\Phi\left(\frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \frac{xe^{-px} + ye^{-py}}{e^{-px} + e^{-py}}\right) = \Phi(x, y), \quad x, y \in \mathbb{R},$$

if, and only if, there is a continuous single variable function  $\varphi: I \rightarrow \mathbb{R}$  such that

$$\Phi(x, y) = \varphi(x + y), \quad x, y \in \mathbb{R}.$$

## 2. Main result

In the proof of the main result we need the following

**LEMMA 1.** *Let  $I \subset \mathbb{R}$  be an interval. Suppose that the functions  $f, g: I \rightarrow (0, \infty)$  are twice differentiable and  $\frac{f}{g}$  is one-to-one. If  $B^{[f,g]}$  is invariant with respect to the mean-type mapping  $(B^{[f]}, B^{[g]})$  i.e., if*

$$B^{[f,g]} \circ (B^{[f]}, B^{[g]}) = B^{[f,g]}, \tag{1}$$

*then there are  $a, r \in \mathbb{R}$ ,  $r \neq 0$ ,  $a > 0$ , such that*

$$f(x) = ae^{rx}g(x), \quad x \in I. \tag{2}$$

Proof. Assume that the functions  $f, g: I \rightarrow (0, \infty)$  satisfy (1). From the definition of the mean  $B^{[f,g]}$  and (1) we have

$$\left(\frac{f}{g}\right)^{-1} \left(\frac{f(B^{[f]}(x, y)) + f(B^{[g]}(x, y))}{g(B^{[f]}(x, y)) + g(B^{[g]}(x, y))}\right) = \left(\frac{f}{g}\right)^{-1} \left(\frac{f(x) + f(y)}{g(x) + g(y)}\right), \quad x, y \in I,$$

whence, for all  $x, y \in I$ ,

$$\begin{aligned} & \left[ f(B^{[f]}(x, y)) + f(B^{[g]}(x, y)) \right] [g(x) + g(y)] \\ &= \left[ g(B^{[f]}(x, y)) + g(B^{[g]}(x, y)) \right] [f(x) + f(y)]. \end{aligned} \quad (3)$$

Differentiation of both sides with respect to  $x$  gives

$$\begin{aligned} & \left[ f'(B^{[f]}) \frac{\partial B^{[f]}}{\partial x} + f'(B^{[g]}) \frac{\partial B^{[g]}}{\partial x} \right] [g(x) + g(y)] + [f(B^{[f]}) + f(B^{[g]})] g'(x) \\ &= \left[ g'(B^{[f]}) \frac{\partial B^{[f]}}{\partial x} + g'(B^{[g]}) \frac{\partial B^{[g]}}{\partial x} \right] [f(x) + f(y)] + [g(B^{[f]}) + g(B^{[g]})] f'(x) \end{aligned}$$

and differentiation with respect to  $y$  gives

$$\begin{aligned} & \left[ f'(B^{[f]}) \frac{\partial B^{[f]}}{\partial y} + f'(B^{[g]}) \frac{\partial B^{[g]}}{\partial y} \right] [g(x) + g(y)] + [f(B^{[f]}) + f(B^{[g]})] g'(y) \\ &= \left[ g'(B^{[f]}) \frac{\partial B^{[f]}}{\partial y} + g'(B^{[g]}) \frac{\partial B^{[g]}}{\partial y} \right] [f(x) + f(y)] + [g(B^{[f]}) + g(B^{[g]})] f'(y), \end{aligned}$$

where  $B^{[f]}$  and  $B^{[g]}$  stand for  $B^{[f]}(x, y)$  and  $B^{[g]}(x, y)$ . Subtracting the respective sides of these equalities and then dividing by  $x - y$  we obtain

$$\begin{aligned} & \left[ f'(B^{[f]}) \frac{\frac{\partial B^{[f]}}{\partial x} - \frac{\partial B^{[f]}}{\partial y}}{x - y} + f'(B^{[g]}) \frac{\frac{\partial B^{[g]}}{\partial x} - \frac{\partial B^{[g]}}{\partial y}}{x - y} \right] \\ & \cdot [g(x) + g(y)] + [f(B^{[f]}) + f(B^{[g]})] \frac{g'(x) - g'(y)}{x - y} \\ &= \left[ g'(B^{[f]}) \frac{\frac{\partial B^{[f]}}{\partial x} - \frac{\partial B^{[f]}}{\partial y}}{x - y} + g'(B^{[g]}) \frac{\frac{\partial B^{[g]}}{\partial x} - \frac{\partial B^{[g]}}{\partial y}}{x - y} \right] \\ & \cdot [f(x) + f(y)] + [g(B^{[f]}) + g(B^{[g]})] \frac{f'(x) - f'(y)}{x - y}. \end{aligned} \quad (4)$$

Since

$$\frac{\partial B^{[f]}}{\partial x} = \frac{f(x)^2 + f(x)f(y) + xf'(x)f(y) - yf'(x)f(y)}{[f(x) + f(y)]^2}$$

and

$$\frac{\partial B^{[f]}}{\partial y} = \frac{f(y)^2 + f(x)f(y) + yf'(y)f(x) - xf'(y)f(x)}{[f(x) + f(y)]^2}$$

it is easy to verify that, for all  $x \in I$ ,

$$\lim_{y \rightarrow x} B^{[f]}(x, y) = \lim_{y \rightarrow x} B^{[g]}(x, y) = x,$$

$$\begin{aligned} & \lim_{y \rightarrow x} \frac{\frac{\partial B^{[f]}}{\partial x} - \frac{\partial B^{[f]}}{\partial y}}{x - y} \\ &= \lim_{y \rightarrow x} \frac{[f(x)^2 + xf'(x)f(y) - yf'(x)f(y)] - [f(y)^2 + yf'(y)f(x) - xf'(y)f(x)]}{[f(x) + f(y)]^2(x - y)} \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

and

$$\lim_{y \rightarrow x} \frac{\frac{\partial B^{[g]}}{\partial x} - \frac{\partial B^{[g]}}{\partial y}}{x - y} = \frac{g'(x)}{g(x)}.$$

Thus, letting  $y \rightarrow x$  in (4), we obtain, for all  $x \in I$ ,

$$\begin{aligned} & 2 \left[ f'(x) \frac{f'(x)}{f(x)} + f'(x) \frac{g'(x)}{g(x)} \right] g(x) + 2f(x)g''(x) \\ &= 2 \left[ g'(x) \frac{f'(x)}{f(x)} + g'(x) \frac{g'(x)}{g(x)} \right] f(x) + 2g(x)f''(x), \end{aligned}$$

whence, after obvious simplification, we get

$$\frac{f''(x)}{f(x)} - \left( \frac{f'(x)}{f(x)} \right)^2 = \frac{g''(x)}{g(x)} - \left( \frac{g'(x)}{g(x)} \right)^2, \quad x \in I.$$

Note that this differential equation, with two unknown functions, can be written in the form

$$\left( \frac{f'}{f} \right)' = \left( \frac{g'}{g} \right)'.$$

It follows that, for some  $r \in \mathbb{R}$ ,

$$\frac{f'}{f} = \frac{g'}{g} + r,$$

whence, for some  $a > 0$ ,

$$\log \circ f(x) = \log \circ g(x) + rx + \log a, \quad x \in I,$$

and, consequently,

$$f(x) = ae^{rx}g(x), \quad x \in I.$$

The injectivity of  $\frac{f}{g}$  implies that  $r \neq 0$ . This completes the proof.  $\square$

In the proof we have obtained the following:

**Remark 1.** Under the assumption of Lemma 1,

$$\left(\frac{f'}{f}\right)' = \left(\frac{g'}{g}\right)'.$$

The main result reads as follows.

**THEOREM 1.** Let  $I \subset \mathbb{R}$  be an interval. Suppose that the functions  $f, g: I \rightarrow (0, \infty)$  are three times differentiable and  $\frac{f}{g}$  is one-to-one. Then the following conditions are equivalent:

(i)  $B^{[f,g]}$  is invariant with respect to the mean-type mapping  $(B^{[f]}, B^{[g]})$ , i.e.,

$$B^{[f,g]} \circ (B^{[f]}, B^{[g]}) = B^{[f,g]};$$

(ii) there are  $a, b, p \in \mathbb{R}$ ,  $p \neq 0$ ,  $a, b > 0$ , such that

$$f(x) = ae^{px}, \quad g(x) = be^{-px}, \quad x \in I;$$

(iii) there is  $p \in \mathbb{R}$ ,  $p \neq 0$ , such that

$$B^{[f,g]}(x, y) = \frac{x + y}{2}, \quad B^{[f]}(x, y) = \frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \quad B^{[g]}(x, y) = \frac{xe^{-px} + ye^{-py}}{e^{-px} + e^{-py}}$$

for all  $x, y \in \mathbb{R}$ .

**Proof.** Assume that  $B^{[f,g]}$  is invariant with respect to the mean-type mapping  $(B^{[f]}, B^{[g]})$ . Then equality (3) is satisfied. Differentiating three times both sides (3) with respect to  $x$  and then letting  $y \rightarrow x$ , we get the equation

$$\left(\frac{g'}{g}\right)^3 - \left(\frac{f'}{f}\right)^3 - \frac{g'}{g} \left(\frac{f'}{f}\right)' - \frac{f'}{f} \left(\frac{g'}{g}\right)' + 2\frac{f'f''}{ff'} - 2\frac{g'g''}{gg'} + \frac{g'''}{g} - \frac{f'''}{f} = 0. \quad (5)$$

From formula (2) of Lemma 1 we have

$$f(x) = ae^{rx}g(x), \quad x \in I,$$

for some  $a, r \in \mathbb{R}$ ,  $r \neq 0$ ,  $a > 0$ , whence, after obvious calculations,

$$\frac{f'}{f} = r + \frac{g'}{g}, \quad \frac{f''}{f} = r^2 + 2r\frac{g'}{g} + \frac{g''}{g}, \quad \frac{f'''}{f} = r^3 + 3r^2\frac{g'}{g} + 3r\frac{g''}{g} + \frac{g'''}{g}.$$

Setting these functions into equation (5) we obtain the differential equation

$$\left(\frac{g'}{g} + r\right) \left(\frac{g'}{g}\right)' = 0.$$

Note that if  $\frac{g'}{g} + r = 0$  on a nonempty open subinterval  $J \subset I$ , then, clearly,  $\left(\frac{g'}{g}\right)' = 0$  on  $J$ . It follows that  $g$  satisfies this differential equation iff

$$\left(\frac{g'}{g}\right)' = 0.$$

Solving this equation we obtain, for some  $b, q \in \mathbb{R}$ ,  $q \neq 0$ ,  $b > 0$ ,

$$g(x) = be^{qx}, \quad x \in I. \tag{6}$$

Since, by Remark 1,

$$\left(\frac{f'}{f}\right)' = 0,$$

we infer that, for some  $a, p \in \mathbb{R}$ ,  $p \neq 0$ ,  $a > 0$ ,

$$f(x) = ae^{px}, \quad x \in I. \tag{7}$$

Substituting the functions  $f$  and  $g$  of the forms (6) and (7) into equation (3), by the analyticity of the involved function, we obtain the equality valid for all  $x, y \in \mathbb{R}$ . Differentiating both sides of this equality with respect to  $x$  and then setting  $x = y = 0$  we get  $p + q = 0$ , which completes the proof of the implication (i)  $\implies$  (ii). Since the remaining implications are easy to verify, the proof is completed.  $\square$

**Remark 2.** Note that the case

$$B^{[f.g]} = B^{[f]} = B^{[g]}$$

cannot happen. Indeed, assume that there are  $f, g: I \rightarrow (0, \infty)$  such that the  $B^{[f]} = B^{[g]}$ . Then

$$\frac{xf(x) + yf(y)}{f(x) + f(y)} = \frac{xg(x) + yg(y)}{g(x) + g(y)}, \quad x, y \in I,$$

whence

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(y)}, \quad x, y \in I.$$

Thus  $\frac{f}{g}$  is a constant function and, consequently, the Bajraktarević mean  $B^{[f.g]}$  does not exist.

**Remark 3.** For every  $p \in \mathbb{R}$ , the function  $\mathcal{B}^{[p]}: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$\mathcal{B}^{[p]}(x, y) := \frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \quad x, y \in \mathbb{R},$$

is a mean, and  $\mathcal{B}^{[0]}$  is the arithmetic mean. Obviously, for every  $p \in \mathbb{R}$ , the mean  $\mathcal{B}^{[p]}$  is of the Beckenbach-Gini type.

### 3. An application

Let us quote the following ([13], [15]):

**THEOREM 2.** *Let  $I \subset \mathbb{R}$  be an interval. If  $M, N: I^2 \rightarrow I$  are continuous means such that at least one of them is strict, then*

- (i) *the sequence of iterates  $((M, N)^n)_{n \in \mathbb{N}}$  of the mean-type mapping  $(M, N): I^2 \rightarrow I^2$  converges to a continuous mean-type mapping  $(K, K): I^2 \rightarrow I^2$  where  $K: I^2 \rightarrow I$  is a continuous mean;*
- (ii)  *$K$  is  $(M, N)$ -invariant, i.e.  $K \circ (M, N) = K$ ;*
- (iii) *a continuous  $(M, N)$ -invariant mean is unique;*
- (iv) *if the means  $M, N$  are strict, then so is  $K$ .*

Now we can prove the following:

**COROLLARY 1.** *Let  $p \in \mathbb{R}$  be fixed.*

- (i) *The arithmetic mean  $A: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $A(x, y) = \frac{x+y}{2}$ , is invariant with respect to the mean-type mapping*

$$\mathbb{R}^2 \ni (x, y) \longrightarrow \left( \frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \frac{xe^{-px} + ye^{-py}}{e^{-px} + e^{-py}} \right);$$

- (ii) *the sequence of iterates of this mapping converges pointwise to the mean-type mapping  $(A, A)$ , that is*

$$\lim_{n \rightarrow \infty} \left( \frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \frac{xe^{-px} + ye^{-py}}{e^{-px} + e^{-py}} \right)^n = \left( \frac{x+y}{2}, \frac{x+y}{2} \right), \quad (x, y) \in \mathbb{R}^2; \tag{7}$$

- (iii) *a function  $\Phi: I^2 \rightarrow \mathbb{R}$ , continuous on the diagonal  $\Delta := \{(x, x) : x \in \mathbb{R}\}$ , satisfies the functional equation*

$$\Phi \left( \frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \frac{xe^{-px} + ye^{-py}}{e^{-px} + e^{-py}} \right) = \Phi(x, y), \quad x, y \in \mathbb{R}, \tag{8}$$

*if, and only if, there is a continuous function in a single variable  $\varphi: I \rightarrow \mathbb{R}$  such that*

$$\Phi(x, y) = \varphi(x + y), \quad x, y \in \mathbb{R}.$$

**Proof.** For  $p = 0$  all parts are obvious.

Assume that  $p \neq 0$ . Part (i) follows from Theorem 1 and part (ii) follows from Theorem 1 and Theorem 2. To prove part (iii), note that from equation



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(7), by induction, we get

$$\Phi(x, y) = \Phi \left( \left( \frac{xe^{px} + ye^{py}}{e^{px} + e^{py}}, \frac{xe^{-px} + ye^{-py}}{e^{-px} + e^{-py}} \right)^n \right), \quad n \in \mathbb{N} \quad x, y \in \mathbb{R}.$$

Letting  $n \rightarrow \infty$  and making use of (7) and the assumed continuity of  $\Phi$  on the diagonal, we obtain

$$\Phi(x, y) = \Phi \left( \frac{x+y}{2}, \frac{x+y}{2} \right), \quad x, y \in \mathbb{R},$$

whence, setting  $\varphi(u) := \Phi(\frac{u}{2}, \frac{u}{2})$  for  $u \in \mathbb{R}$ , we get the desired form of  $\Phi$ . Since the converse implication is easy to verify, the proof is completed.  $\square$

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