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Fixed point theorem in a uniformly convex paranormed space and its application

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ABSTRACT

For a measure space (Ω, Σ, μ) with $\mu(\Omega) \leqslant 1$, under some general conditions on a bijective function $\varphi : [0, \infty) \to [0, \infty)$, a family of μ-integrable functions $x : \Omega \to \mathbb{R}$ with the functional $\mathbf{p}_\varphi$ defined by

$$\mathbf{p}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x|\, d\mu\right),$$

forms a paranormed uniformly convex space $(\mathcal{S}^\varphi(\Omega, \Sigma, \mu), \mathbf{p}_\varphi)$ (an extension of L^p space). Applying a generalization of the Browder–Goehde–Kirk-type fixed point theorem due to Pasicki, we present sufficient conditions for existence of a solution $x \in \mathcal{S}^\varphi(\Omega, \Sigma, \mu)$ of a nonlinear functional equation. Moreover some new fixed results are proved.

1. Introduction

Given a measure space (Ω, Σ, μ) denote by $S = S(\Omega, \Sigma, \mu)$ the linear real space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. For a bijection $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$, define the functional $\mathbf{p}_\varphi : S \to [0, \infty)$ by

$$\mathbf{p}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x|\, d\mu\right).$$

In [6] we have shown that if $\mu(\Omega) \leqslant 1$, the function φ is super-quadratic i.e., if

$$\varphi(r+s) + \varphi\big(|r-s|\big) \geqslant 2\varphi(r) + 2\varphi(s), \quad r, s \geqslant 0,$$

and the function $(r, s) \longmapsto \varphi(\varphi^{-1}(r) + \varphi^{-1}(s))$ is concave, then $(S(\Omega, \Sigma, \mu), \mathbf{p}_\varphi)$ (as well as its completion $(\mathcal{S}^\varphi(\Omega, \Sigma, \mu), \mathbf{p}_\varphi)$) is a uniformly convex paranormed space (F-space) (Section 2). It turns out that this space has a nice geometrical property; namely, it is a bead space in a sense of Pasicki [13] (cf. also [14,15]). In Section 3, applying a generalization of the Browder–Goehde–Kirk fixed point theorem due to Pasicki [12,13], we give conditions under which the functional equation

$$x(\tau) = h\big(\tau, x\big(f(\tau)\big)\big)$$

has a solution $x \in \mathcal{S}^\varphi(\Omega, \Sigma, \mu)$. In Section 4 we show that in the Menger convex metric space, the nonexpansivity assumption in Pasicki's fixed theorem can be replaced by formally a much weaker one (Theorem 5). Assume that $(X, \mathbf{p})$ is a

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complete and uniformly convex paranormed space with a continuous modulus of convexity; $C \subset X$ is a nonempty, bounded, closed and convex; a mapping $T : C \to C$ is radially continuous at all points except for at most one; and there is a function $\gamma : [0, \infty) \to [0, \infty)$ such that

$$\mathbf{p}\big(T(x) - T(y)\big) \leqslant \gamma\big(\mathbf{p}(x-y)\big), \quad x, y \in C,$$

and

$$\liminf_{t\to 0+} \frac{\gamma(t)}{t} \leqslant 1.$$

Applying Theorem 5 we conclude that T has a fixed point. This is a new generalization of the Browder–Goehde–Kirk fixed point theorem [3,4].

2. Uniformly convex paranormed space, bead space and a fixed point of Pasicki

Let X be a real linear space. A function $\mathbf{p} : X \to \mathbb{R}$ is called a *paranorm* (a *total paranorm*, Wilansky [16, p. 52]; or *F-norm*, Musielak [11, p. 62]) on X if $\mathbf{p}(x) = 0$ iff $x = \mathbf{0}$, $\mathbf{p}(-x) = \mathbf{p}(x)$ for all $x \in X$, $\mathbf{p}$ is subadditive, i.e.

$$\mathbf{p}(x+y) \leqslant \mathbf{p}(x) + \mathbf{p}(y), \quad x, y \in X;$$

and, if $t_n, t \in \mathbb{R}$, $x_n, x \in X$ for $n \in \mathbb{N}$, are such that $t_n \to t$, $\mathbf{p}(x_n - x) \to 0$, then $\mathbf{p}(t_n x_n - tx) \to 0$. If $\mathbf{p}$ is a paranorm then $(X, \mathbf{p})$ is called a *paranormed space*. If $(X, \mathbf{p})$ is a paranormed space then, obviously, (X, d) with $d(x, y) := \mathbf{p}(x-y)$ is a metric space. If this metric space is complete, we say that the paranormed space is complete.

We say that a paranormed space $(X, \mathbf{p})$ is *uniformly convex* if for every $r > 0$ and $\varepsilon \in (0, 2r)$ there exists $\delta(r, \varepsilon) \in (0, r)$ such that, for all $x, y \in X$,

$$\big[\mathbf{p}(x) \leqslant r \wedge \mathbf{p}(y) \leqslant r \wedge \mathbf{p}(x-y) \geqslant \varepsilon\big] \quad \Longrightarrow \quad \mathbf{p}\left(\frac{x+y}{2}\right) \leqslant r - \delta(r, \varepsilon).$$

The function $\delta : \Delta \to (0, \infty)$, where $\Delta := \{(r, \varepsilon): \ r > 0, \ 0 < \varepsilon < 2r\}$, is referred to as the *modulus of convexity* of the space $(X, \mathbf{p})$.

Let (Ω, Σ, μ) be a measure and $S = S(\Omega, \Sigma, \mu)$ the linear real space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. Let $S_+ := \{x \in S: \ x \geqslant 0\}$. If $\varphi : [0, \infty) \to [0, \infty)$ is bijective and $\varphi(0) = 0$, then the functional $\mathbf{p}_\varphi : S \to [0, \infty)$ given by the formula

$$\mathbf{p}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x| \, d\mu\right), \quad x \in S(\Omega, \Sigma, \mu),$$

is correctly defined. If $x \in S$ then $x = \sum_{i=1}^k r_i \chi_{A_i}$ for some $k \in \mathbb{N}$, $r_i \in \mathbb{R}$, and the pairwise disjoined sets $A_i \in \Sigma$, $i = 1, \dots, k$; moreover,

$$\mathbf{p}_\varphi(x) = \varphi^{-1}\left(\sum_{i=1}^k \varphi\big(|r_i|\big)\mu(A_i)\right).$$

(Here χ_A denotes the characteristic function of the set A.)

From [8, Theorem 6(a)] (cf. also Hardy, Littlewood, Pólya [5, Theorem 106(ii)]) we have the following:

Lemma 1. *Let (Ω, Σ, μ) be a measure space and $\varphi : [0, \infty) \to [0, \infty)$ be an increasing bijection.*

(a) *Suppose that $\mu(\Omega) = 1$ and there exists a set $A \in \Sigma$ such that $0 < \mu(A) < 1$. Then $\mathbf{p}_\varphi$ is a paranorm in $S(\Omega, \Sigma, \mu)$ if, and only if, the function $F : [0, \infty)^2 \to [0, \infty)$,*

$$F(r, s) := \varphi\big(\varphi^{-1}(r) + \varphi^{-1}(s)\big) \tag{1}$$

is concave.

(b) *If $\mu(\Omega) \leqslant 1$ and the function F is concave, then $\mathbf{p}_\varphi$ is a paranorm in $S(\Omega, \Sigma, \mu)$.*

Remark 1. Let $\varphi : [0, \infty) \to [0, \infty)$ be bijective and increasing and such that the function (1) is concave. Then φ is continuous, $\varphi(0) = 0$, and the functional $\mathbf{p}_\varphi : S \to [0, \infty)$ has the following properties: $\mathbf{p}_\varphi(x) = 0$ for an $x \in S$ iff $x = 0$ μ-a.e.; $\mathbf{p}_\varphi(x) = \mathbf{p}_\varphi(-x)$ for all $x \in S$; if $t_n, t \in \mathbb{R}$, $x_n, x \in S$ for $n \in \mathbb{N}$, are such that $t_n \to t$, $\mathbf{p}_\varphi(x_n - x) \to 0$, then $\mathbf{p}_\varphi(t_n x_n - tx) \to 0$ (cf. [8], proof of Theorem 13).

Remark 2. ([7]) Let $\varphi : [0, \infty) \to [0, \infty)$ be a twice differentiable function such that $\varphi(0) = 0$, $\varphi'(r) > 0$ and $\varphi''(r) > 0$ for all $r > 0$.

If the function $\frac{\varphi'}{\varphi''}$ is superadditive in $(0,\infty)$, i.e.

$$\frac{\varphi'(r+s)}{\varphi''(r+s)} \geqslant \frac{\varphi'(r)}{\varphi''(r)} + \frac{\varphi'(s)}{\varphi''(s)}, \quad r,s>0,$$

the function (1) is concave in $[0,\infty)^2$.

Since the lemma below, as well as the following two theorems, have been proved in the paper [6] that is not yet published, we present them with their proofs.

Lemma 2. *Let (Ω,Σ,μ) be a measure space such that $\mu(\Omega)\leqslant 1$. Let $\varphi:[0,\infty)\to[0,\infty)$ be bijective, $\varphi(0)=0$ and let the function $F:[0,\infty)^2\to[0,\infty)$ be defined by*

$$F(r,s) := \varphi\big(\varphi^{-1}(r)+\varphi^{-1}(s)\big), \quad r,s\geqslant 0.$$

If F is concave, then $\mathbf{p}_\varphi$ is a paranorm in $S(\Omega,\Sigma,\mu)$. If moreover φ is super-quadratic, i.e.,

$$\varphi(r+s)+\varphi\big(|r-s|\big) \geqslant 2\varphi(r)+2\varphi(s), \quad r,s\geqslant 0, \tag{2}$$

then

$$\varphi\big(\mathbf{p}_\varphi(x+y)\big)+\varphi\big(\mathbf{p}_\varphi(x-y)\big) \geqslant 2\varphi\big(\mathbf{p}_\varphi(x)\big)+2\varphi\big(\mathbf{p}_\varphi(y)\big), \quad x,y\in S(\Omega,\Sigma,\mu).$$

Proof. Assume first that $\mu(\Omega)=1$. Take arbitrary $x,y\in S(\Omega,\Sigma,\mu)$. Then

$$x=\sum_{i=1}^k r_i\chi_{A_i}, \qquad y=\sum_{i=1}^k s_i\chi_{A_i}$$

for some $k\in\mathbb{N}$, $r_i,s_i\in\mathbb{R}$, and the pairwise disjoined sets $A_i\in\Sigma$, $i=1,\dots,k$, such that $\sum_{i=1}^k\mu(A_i)=1$. Without any loss of generality, we can assume that all r_i, s_i are non-negative and such that $s_i\leqslant r_i$ for $i=1,\dots,k$. Put $a_i=\mu(A_i)$, $i=1,\dots,k$. From (2) we have

$$\big[\varphi(r_i+s_i)+\varphi(r_i-s_i)\big]a_i \geqslant 2\varphi(r_i)a_i+2\varphi(s_i)a_i, \quad i=1,\dots,k,$$

whence

$$\sum_{i=1}^k\varphi(r_i+s_i)a_i+\sum_{i=1}^k\varphi(r_i-s_i)a_i \geqslant 2\sum_{i=1}^k\varphi(r_i)a_i+2\sum_{i=1}^k\varphi(s_i)a_i,$$

or, equivalently,

$$\varphi\left[\varphi^{-1}\left(\sum_{i=1}^k\varphi(r_i+s_i)a_i\right)\right]+\varphi\left[\varphi^{-1}\left(\sum_{i=1}^k\varphi(r_i-s_i)a_i\right)\right] \geqslant 2\varphi\left[\varphi^{-1}\left(\sum_{i=1}^k\varphi(r_i)a_i\right)\right]+2\varphi\left[\varphi^{-1}\left(\sum_{i=1}^k\varphi(s_i)a_i\right)\right],$$

which means that

$$\varphi\big(\mathbf{p}_\varphi(x+y)\big)+\varphi\big(\mathbf{p}_\varphi(x-y)\big) \geqslant 2\varphi\big(\mathbf{p}_\varphi(x)\big)+2\varphi\big(\mathbf{p}_\varphi(y)\big),$$

which was to be shown. □

Theorem 1. *Let (Ω,Σ,μ) be a measure space and $\varphi:[0,\infty)\to[0,\infty)$ be an increasing bijection such that $\mathbf{p}_\varphi$ is a paranorm in $S(\Omega,\Sigma,\mu)$. If the function φ is super-quadratic, then the paranormed space $(S(\Omega,\Sigma,\mu),\mathbf{p}_\varphi)$ is uniformly convex, and the function*

$$\delta(r,\varepsilon)=r-\varphi^{-1}\left(\varphi(r)-\varphi\left(\frac{\varepsilon}{2}\right)\right), \quad r>0,\ \varepsilon\in(0,2r),$$

is a modulus convexity of this space.

Proof. Take arbitrary $r>0$, $\varepsilon\in(0,2r)$ and $x,y\in S(\Omega,\Sigma,\mu)$ such that

$$\mathbf{p}_\varphi(x)\leqslant r, \qquad \mathbf{p}_\varphi(y)\leqslant r \quad \text{and} \quad \mathbf{p}_\varphi(x-y)\geqslant\varepsilon.$$

Putting

$$x_1 := \frac{x+y}{2}, \qquad y_1 := \frac{x-y}{2}$$

we have

$$x = x_1 + y_1, \qquad y = x_1 - y_1, \qquad \mathbf{p}_\varphi(y_1) \geqslant \frac{\varepsilon}{2}$$

(as $\varepsilon \leqslant \mathbf{p}_\varphi(x-y) = \mathbf{p}_\varphi(2y_1) \leqslant 2\mathbf{p}_\varphi(y_1)$), and

$$\varphi\big(\mathbf{p}_\varphi(x_1+y_1)\big) + \varphi\big(\mathbf{p}_\varphi(x_1-y_1)\big) \leqslant 2\varphi(r).$$

From the "moreover" part of Lemma 2 we have

$$2\varphi\big(\mathbf{p}_\varphi(x_1)\big) + 2\varphi\big(\mathbf{p}_\varphi(y_1)\big) \leqslant \varphi\big(\mathbf{p}_\varphi(x_1+y_1)\big) + \varphi\big(\mathbf{p}_\varphi(x_1-y_1)\big) \leqslant 2\varphi(r).$$

Hence, as φ is increasing,

$$\varphi\big(\mathbf{p}_\varphi(x_1)\big) \leqslant \varphi(r) - \varphi\big(\mathbf{p}_\varphi(y_1)\big) \leqslant \varphi(r) - \varphi\left(\frac{\varepsilon}{2}\right),$$

whence

$$\mathbf{p}_\varphi\left(\frac{x+y}{2}\right) = \mathbf{p}_\varphi(x_1) \leqslant \varphi^{-1}\left(\varphi(r) - \varphi\left(\frac{\varepsilon}{2}\right)\right).$$

Setting

$$\delta = \delta(r,\varepsilon) := r - \varphi^{-1}\left(\varphi(r) - \varphi\left(\frac{\varepsilon}{2}\right)\right),$$

we obtain

$$\mathbf{p}_\varphi\left(\frac{x+y}{2}\right) \leqslant r - \delta,$$

which completes the proof. □

From this result and Lemma 2 we immediately obtain [6].

Theorem 2. *Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) \leqslant 1$. If $\varphi : [0,\infty) \to [0,\infty)$ is an increasing bijection and the function*

$$[0,\infty)^2 \ni (s,r) \longmapsto \varphi\big(\varphi^{-1}(r) + \varphi^{-1}(s)\big)$$

is concave, then $\mathbf{p}_\varphi$ is a paranorm in $(S(\Omega,\Sigma,\mu), \mathbf{p}_\varphi)$. If moreover φ is super-quadratic, then the paranormed space $(S(\Omega,\Sigma,\mu), \mathbf{p}_\varphi)$ is uniformly convex, and the function

$$\delta(r,\varepsilon) = r - \varphi^{-1}\left(\varphi(r) - \varphi\left(\frac{\varepsilon}{2}\right)\right), \quad r > 0,\ \varepsilon \in (0, 2r),$$

is its modulus of convexity.

Remark 3. If the paranormed space $(S(\Omega,\Sigma,\mu), \mathbf{p}_\varphi)$ is uniformly convex, then, obviously, its completion denoted by $(\mathcal{S}^\varphi(\Omega,\Sigma,\mu), \mathbf{p}_\varphi)$, is also uniformly convex (with the same modulus of convexity). Taking $\varphi(t) = t^p$ $(t \geqslant 0)$ with $p \geqslant 1$ in Theorem 1 we obtain the Clarkson theorem on the uniform convexity of the L^p spaces [2].

Following Pasicki [13–15], a metric space (X,d) is said to be a *bead space* if for every $r > 0$, $\beta > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $d(x,y) \geqslant \beta$ there exists $z \in X$ such that $B(x, r+\delta) \cap B(y, r+\delta) \subset B(z, r-\delta)$. (Here $B(x,r)$ denotes the open ball with the center x and the radius r.)

Lemma 3. *([15]) Let Z be a linear space (or a set in a linear space such that $Z - Z \subset Z$, $2X \subset Z$) and let $\mathbf{p} : Z \to \mathbb{R}$ be a function such that $d(x,y) := \mathbf{p}(x-y)$, $x, y \in X$ defines a metric on a set $X \subset Z$ satisfying $(X+X)/2 \subset X$. Assume that the following condition holds:*

For every $r > 0$, $\beta > 0$ there exists $\delta > 0$ such that for each $x, y \in Z$,

$$\big[\mathbf{p}(x) < r + \delta \wedge \mathbf{p}(y) < r + \delta \wedge \mathbf{p}(x-y) > \beta\big] \quad \Longrightarrow \quad \mathbf{p}\left(\frac{x+y}{2}\right) \leqslant r - \delta.$$

Then the metric space (X,d) is a bead space.

This lemma implies that each uniformly continuous paranormed space is a bead space. Moreover, from Pasicki [12–15] we have the following fixed point theorem:

Theorem 3. *Let $(X, \mathbf{p})$ be a complete and uniformly convex paranormed space with a continuous modulus of convexity. Let $C \subset X$ be a nonempty, bounded, closed and convex. If $T : C \to C$ is nonexpansive i.e., if*

$$\mathbf{p}(Tx - Ty) \leqslant \mathbf{p}(x - y), \quad x, y \in C,$$

then T has a fixed point.

3. An application

In this section we present an application of Theorem 3 in the theory of functional equations. Put $I = [0, 1]$.

Theorem 4. *Let (Ω, Σ, μ) be a measure space such that $\Omega = I$, Σ is the σ-algebra of all Lebesgue measurable subsets of I, and μ the Lebesgue measure restricted to Σ. Let a bijective function $\varphi : [0, \infty) \to [0, \infty)$ be increasing, convex, super-quadratic, and such that the function*

$$(s, t) \longmapsto \varphi\big(\varphi^{-1}(s) + \varphi^{-1}(t)\big)$$

is concave.

Suppose that $f : I \to I$ is increasing and differentiable, $h : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory conditions and there are $\alpha, \beta \in \mathcal{S}^\varphi(\Omega, \Sigma, \mu)$ such that

$$\alpha(\tau) \leqslant h\big(\tau, \alpha(\tau)\big) \leqslant h\big(\tau, \beta(\tau)\big) \leqslant \beta(\tau), \quad \tau \in [0, 1]; \tag{3}$$

for all $\tau \in [0, 1]$ and $x \in [\alpha(\tau), \beta(\tau)]$, we have

$$h\big(\tau, \alpha(\tau)\big) \leqslant h(\tau, x) \leqslant h\big(\tau, \beta(\tau)\big); \tag{4}$$

there exists a Lebesgue measurable function $g : I \to [0, \infty)$ such that

$$\big|h(\tau, x) - h(\tau, y)\big| \leqslant g(\tau)|x - y|, \quad \tau \in I, \ x, y \in \big[\alpha(\tau), \beta(\tau)\big]. \tag{5}$$

If moreover

$$\frac{g}{f'} \leqslant 1 \tag{6}$$

then the functional equation

$$x(\tau) = h\big(\tau, x\big(f(\tau)\big)\big)$$

has at least one solution $x \in \mathcal{S}^\varphi(\Omega, \Sigma, \mu)$.

Proof. Put

$$C := \big\{x \in \mathbb{R}^I \colon \alpha(\tau) \leqslant x(\tau) \leqslant \beta(\tau)\big\} \cap \mathcal{S}^\varphi(\Omega, \Sigma, \mu).$$

Of course C is a closed convex and bounded subset of $\mathcal{S}^\varphi(\Omega, \Sigma, \mu)$. Define $T : C \to \mathbb{R}^I$ by

$$T(x)(\tau) := h\big(\tau, x\big(f(\tau)\big)\big), \quad x \in C.$$

We shall show that T maps C into itself. Indeed, for $x \in \mathcal{S}^\varphi(\Omega, \Sigma, \mu)$, the function $x \circ f$ is measurable and, by the Caratheodory conditions, so is the function $T(x)$. Since $\alpha(\tau) \leqslant x(\tau) \leqslant \beta(\tau)$ for all $\tau \in I$, the conditions (3), (4) imply that

$$\alpha(\tau) \leqslant T(x)(\tau) \leqslant \beta(\tau), \quad \tau \in I,$$

which proves that $T(x) \in C$. For $x, y \in C$, by the definition of $\mathbf{p}_\varphi$ and conditions (5), (6) and the increasing monotonicity of φ, we have

$$\begin{aligned}
\mathbf{p}_\varphi\big(T(x) - T(y)\big) &= \varphi^{-1}\left(\int_0^1 \varphi\big(\big|\big(T(x) - T(y)\big)(\tau)\big|\big)\, d\tau\right) \\
&= \varphi^{-1}\left(\int_0^1 \varphi\big(\big|h\big(\tau, x\big(f(\tau)\big)\big) - h\big(\tau, y\big(f(\tau)\big)\big)\big|\big)\, d\tau\right) \\
&\leqslant \varphi^{-1}\left(\int_0^1 \varphi\big(g(\tau)\big|x\big(f(\tau)\big) - y\big(f(\tau)\big)\big|\big)\, d\tau\right)
\end{aligned}$$

$$= \varphi^{-1}\left(\int_0^1 \varphi\left(\frac{g(\tau)}{f'(\tau)}\left|x\big(f(\tau)\big) - y\big(f(\tau)\big)\right| f'(\tau)\right) d\tau\right)$$

$$\leqslant \varphi^{-1}\left(\int_0^1 \varphi\big(\left|x\big(f(\tau)\big) - y\big(f(\tau)\big)\right| f'(\tau)\big)\, d\tau\right)$$

$$= \varphi^{-1}\left(\int_{f(0)}^{f(1)} \varphi\big(\left|x(\tau) - y(\tau)\right|\big)\, d\tau\right)$$

$$\leqslant \varphi^{-1}\left(\int_0^1 \varphi\big(\left|x(\tau) - y(\tau)\right|\big)\, d\tau\right) = \mathbf{p}_\varphi(x - y),$$

so T is nonexpansive. Now the result follows from the fixed point Theorem 3. □

Remark 4. Inequality (6) can be assumed to hold μ-almost everywhere. (In particular f' can be equal zero on a set of measure zero.)

4. Menger convex metric space, pointwise radially continuous function, Lipschitz map and a fixed point theorem

A metric space (X, d) is called *Menger convex* if for any $x, y \in X$, $x \neq y$, there is $z \in X$, $x \neq z \neq y$, such that

$$d(x, y) = d(x, z) + d(z, y).$$

We have the following (Blumenthal [1]):

Lemma 4. *Let (X, d) be a Menger convex metric space. Then for any pair of points $x, y \in X$ there exists at least one metric segment $[x, y]$ of the endpoints x and y, i.e., for every $x, y \in X$, $x \neq y$, there exists a function $m : [0, d(x, y)] \to X$ such that*

$$m(0) = x, \qquad m\big(d(x, y)\big) = y,$$

and, for every $s, t \in [0, d(x, y)]$,

$$d\big(m(s), m(t)\big) = |s - t|;$$

in particular, for every $x, y \in X$ and $\alpha \in (0, 1)$ there exists $z \in X$ such that

$$d(x, z) = \alpha d(x, y), \qquad d(z, y) = (1 - \alpha) d(x, y).$$

Definition 1. Let (X, d) be a Menger convex metric space and let (Y, ρ) be a metric space. We say that a mapping $T : X \to Y$ is radially continuous at a point $x \in X$ if, for every $y \in X$, and for any metric segment $[x, y]$, the restriction $T|_{[x,y]}$ is continuous at x.

Proposition 1. *Let (X, d) and be (Y, ρ) be a metric spaces and (X, d) be Menger convex. Suppose that $T : X \to Y$ is radially continuous at all points except for at most one. If there exist a real $c \geqslant 0$ and a sequence of positive real numbers (t_n), $\lim_{n\to\infty} t_n = 0$, such that*

$$d(x, y) = t_n \quad \Longrightarrow \quad \rho\big(T(x), T(y)\big) \leqslant c t_n, \tag{7}$$

for all $n \in \mathbb{N}$, $x, y \in X$, then

$$\rho\big(T(x), T(y)\big) \leqslant c d(x, y), \quad x, y \in X.$$

Proof. Put

$$A := \big\{t \geqslant 0\colon\ \forall x, y \in X,\ d(x, y) = t \Longrightarrow \rho\big(T(x), T(y)\big) \leqslant ct\big\}.$$

If $t \geqslant \operatorname{diam} X$ and $t < \infty$ then, of course, $t \in A$. Moreover, by the assumption,

$$t_n \in A, \quad n \in \mathbb{N}. \tag{8}$$

Let $x, y \in X$ be such that, for some $k, n \in \mathbb{N}$,

$$d(x, y) = k t_n.$$

By Lemma 4 there is a segment $[x, y]$ and the points $z_j \in [x, y]$ such that

$$d(x, z_j) = \frac{j}{k} d(x, y), \quad j = 0, 1, \ldots, k,$$

in particular we have

$$z_0 = x, \qquad z_k = y; \qquad d(z_j, z_{j-1}) = t_n, \quad j = 1, 2, \ldots, k.$$

Hence, making use of (8),

$$\rho\big(T(x), T(y)\big) \leqslant \sum_{j=1}^{k} \rho\big(T(z_j), T(z_{j-1})\big) \leqslant c \sum_{j=1}^{k} \|z_j - z_{j-1}\| = ckt_n.$$

This proves that $kt_n \in A$ for all $k, n \in \mathbb{N}$. Since the set $\{kt_n\colon k, n \in \mathbb{N}\}$ is dense in $[0, +\infty)$, we infer that so is A.

Now take arbitrary $x, y \in X$, $x \neq y$, and put

$$t = d(x, y).$$

Without any loss of generality we can assume that T is radially continuous at x. By the density of A there is a sequence (s_n) of real numbers such that

$$s_n \in A, \quad 0 < s_n < t \quad \text{for all } n \in \mathbb{N}; \qquad \lim_{n\to\infty} s_n = t.$$

Take $x_n \in [x, y]$ such that

$$d(x, x_n) = \left(1 - \frac{s_n}{t}\right) d(x, y), \quad n \in \mathbb{N}.$$

Of course we have

$$\lim_{n\to\infty} x_n = x.$$

Since $s_n \in A$ we have

$$\rho\big(T(x_n), T(y)\big) \leqslant cd(x_n, y), \quad n \in \mathbb{N}.$$

From the assumed radial continuity of T at x, letting $n \to \infty$, we hence get

$$\rho\big(T(x), T(y)\big) \leqslant cd(x, y),$$

which was to be shown. □

Corollary 1. *Let (X, d) be a metric space that is convex in the sense of Menger and let (Y, ρ) be a metric space. Suppose $T : X \to Y$ is radially continuous at all points except for at most one. If there are two sequences of positive real numbers (c_n) and (t_n), $\lim_{n\to\infty} t_n = 0$, such that for all $n \in \mathbb{N}$, $x, y \in X$,*

$$d(x, y) = t_n \quad \Longrightarrow \quad \rho\big(T(x), T(y)\big) \leqslant c_n t_n, \tag{9}$$

then

$$\rho\big(T(x), T(y)\big) \leqslant cd(x, y), \quad x, y \in X,$$

where $c = \liminf_{n\to\infty} c_n$.

Proof. If $c = +\infty$ there is nothing to prove. Assume that $0 \leqslant c < +\infty$. For arbitrary $\varepsilon > 0$ there exists a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ such that $c_{n_k} \leqslant c + \varepsilon$ for all $k \in \mathbb{N}$. Hence, by (9), for all $n \in \mathbb{N}$, $x, y \in X$, we have

$$d(x, y) = t_{n_k} \quad \Longrightarrow \quad \rho\big(T(x), T(y)\big) \leqslant (c + \varepsilon) t_{n_k}.$$

In view of Proposition 1 we obtain that $\rho(T(x), T(y)) \leqslant (c + \varepsilon) d(x, y)$ for all $x, y \in X$. Letting $\varepsilon \to 0$ we obtain the result. □

The following example shows that the assumption of the radial continuity of the mapping T cannot be omitted.

Example 1. Let $X = Y = C = \mathbb{R}$, and let $T : C \to \mathbb{R}$ be defined by

$$T(x) := \begin{cases} x + 1 & \text{for } x \in \mathbb{Q}, \\ x + 2 & \text{for } x \notin \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}$ denotes the set of rational numbers. Then, for all $x, y \in C$,

$$|x - y| \in \mathbb{Q} \quad \Longrightarrow \quad \big|T(x) - T(y)\big| = |x - y|.$$

In particular, with every sequence of positive rational numbers (t_n), such that $\lim_{n\to\infty} t_n = 0$, the mapping T satisfies condition (7).

From Corollary 1 we obtain the following generalization of the Browder–Goehde–Kirk fixed point theorem.

Theorem 5. *Let $(X, \mathbf{p})$ be a complete and uniformly convex paranormed space with a continuous modulus of convexity. Let $C \subset X$ be a nonempty, bounded, closed and convex. Suppose $T : C \to C$ is radially continuous at all points except for at most one. If there are two sequences of positive real numbers (c_n) and (t_n), $\lim_{n\to\infty} t_n = 0$, such that for all $n \in \mathbb{N}$, $x, y \in C$,*

$$\mathbf{p}(x-y) = t_n \quad \Longrightarrow \quad \mathbf{p}\big(T(x) - T(y)\big) \leqslant c_n t_n,$$

and

$$\liminf_{n\to\infty} c_n \leqslant 1,$$

then T has a fixed point.

Proof. Applying Corollary 1 with $X = Y = C$, $d(x, y) := \mathbf{p}(x-y)$ for $x, y \in C$, we obtain

$$\mathbf{p}(Tx - Ty) \leqslant \mathbf{p}(x-y), \quad x, y \in C,$$

that is nonexpansive and the result follows from Theorem 3. □

Remark 5. For $c_n = 1$ for all $n \in \mathbb{N}$ we get the main result of [9]. Under the assumption of the continuity of T, the respective result is proved in [10].

Corollary 3. *Let $(X, \mathbf{p})$ be a complete and uniformly convex paranormed space with a continuous modulus of convexity. Let $C \subset X$ be a nonempty, bounded, closed and convex. Suppose $T : C \to C$ is radially continuous at all points except for at most one. If there is a function $\gamma : [0, \infty) \to [0, \infty)$ such that*

$$\mathbf{p}\big(T(x) - T(y)\big) \leqslant \gamma\big(\mathbf{p}(x-y)\big), \quad x, y \in C,$$

and

$$\liminf_{t\to 0+} \frac{\gamma(t)}{t} \leqslant 1,$$

then T has a fixed point.

Remark 6. Taking here a uniformly convex Banach space for X and a function $\gamma(t) = t$ $(t \geqslant 0)$ we obtain the Browder–Goehde–Kirk fixed point theorem (cf. [3,4]).

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