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Results of Tadeusz Świątkowski on algebraic sums of sets and their applications in the theory of subadditive functions

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Dedicated to the memory of Tadeusz Świątkowski

Abstract. A refinement of Steinhaus' theorem on the algebraic sum of subsets of $\mathbb{R}$ due to Raikov (1939) was not known to the mathematical community and still is not popular. In 1994, Tadeusz Świątkowski, being not aware of the existence of Raikov's theorem, proved another result of this type. Unfortunately, a few days later he passed away. In this paper we present the theorems of Świątkowski and Raikov and we apply them in the theory of subadditive type inequalities. An improvement of a converse of Minkowski's inequality theorem is presented.

Keywords. Algebraic sum of sets, density of set at a point, interior point, Steinhaus theorem, Raikov theorem, subadditive function.

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1 Introduction

Let m denote the inner Lebesgue measure in $\mathbb{R}$. The celebrated theorem of Hugo Steinhaus asserts that if $A, B \subset \mathbb{R}$ are arbitrary sets such that $m(A) > 0$ and $m(B) > 0$, then the set $A + B := \{a + b : a \in A, b \in B\}$ has a nonempty interior [13]. This result, as well as its generalizations, are useful in the theory of additive functions and convex functions (cf. [3]). The theorem of Steinhaus is a crucial tool in the proofs (i) that every additive function bounded from above (or from below) on a set of positive measure is linear, via the Bernstein–Doetsch theorem [1] (cf. also [3, p. 145]), (ii) that every Jensen convex function bounded from above on a set of positive measure is convex (theorem of Ostrowski [8]), and (iii) that any Jensen convex measurable function is convex (theorem of Sierpiński [12]).

In the regularity theory of subadditive functions the following question is interesting. Let $A \subset \mathbb{R}$ be such that $\lambda_0(A)$, the measure of density of the set A at point

0, is positive. Does there exist a positive integer n such that

$$0 \in \operatorname{int} \sum_{j=1}^{n} A$$

or, for some $\delta > 0$,

$$(0, \delta) \subset \sum_{j=1}^{n} A?$$

Note that the theorem of Steinhaus does not indicate a point which belongs to the interior of the algebraic sum of the involved sets.

This question was discussed with Professor Tadeusz Świątkowski in February 1994 as we were interested in subadditive functions [5–7]. During the 32nd International Symposium on Functional Equations (ISFE), June 1994, it was posed as an open problem. I was completely surprised when a while later, in a private discussion, the German mathematician Wolfgang Sander suggested that I should contact Professor Świątkowski as somebody who might be helpful in solving this problem! So just after this meeting, in a phone call, I told this story to Tadeusz. A few months later he had to undergo a by-pass operation. Just before going to hospital he gave me a sketch of a solution of the problem. Unfortunately, a few days later, on October 30, 1994, Tadeusz passed away.

After some time I submitted a joint paper on subadditive functions where the results of Tadeusz were included. About one year later the paper was rejected. It turned out that the problem had already been solved in 1939 by Raikov [11].

In these circumstances it is interesting that mathematicians working in convex or subadditive functions for more than half of the century were not aware of the existence of the result of Raikov. Its appearance in the year when the second world war began partially justifies this fact. Let us also remark that Raikov in [11] does not mention the theorem of Steinhaus published in 1919 in Fundamenta Mathematica.

In Section 2 of this paper we present the solution of the problem, based on the idea of Tadeusz Świątkowski (different than that of Raikov) and some accompanying results. Raikov's theorem and its consequence are presented in Section 3.

Using these theorems, in Section 4, we prove some new results on the continuity of a function $f : (0, \infty) \to \mathbb{R}$ satisfying, for some positive α and β, the inequality

$$f(x + \alpha y) \le f(x) + \beta f(y), \quad x, y > 0.$$

Applying a special case when $\alpha = \beta = 1$ we prove an improvement of the converse of Minkowski's inequality theorem [4].

2 Results of Tadeusz Świątkowski

Let $A \subset \mathbb{R}$ be an arbitrary set. Recall that $m(A)$, the inner Lebesgue measure of A, is defined by

$$m(A) := \sup\{l_1(F) : F \subset A,\ F \text{ closed}\},$$

where l_1 is the Lebesgue measure.

For $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, by $\lambda_a(A), \lambda_a^+(A), \lambda_a^-(A)$ denote, respectively, the lower density, the right-lower density and the left-lower density of the set A at the point a, defined by the formulas

$$\lambda_a(A) := \liminf_{h\to 0+} \frac{m(A \cap [a-h, a+h])}{2h},$$

$$\lambda_a^+(A) := \liminf_{h\to 0+} \frac{m(A \cap [a, a+h])}{h}, \quad \lambda_a^-(A) := \liminf_{h\to 0+} \frac{m(A[a-h, a])}{h}.$$

We say that a is a density point of a measurable set A when $\lambda_a(A) = 1$. The most important property of the density points of A is described in the famous Lebesgue density theorem which says that the set of those points of A which are not density points of A has measure zero (see, e.g., [9]).

The following properties are easy to verify:

Remark 1. For all $A \subset \mathbb{R}$, $a \in \mathbb{R}$ and $t > 0$,

$$\lambda_a(A), \lambda_a^+(A), \lambda_a^-(A) \in [0, 1], \quad \frac{1}{2}[\lambda_a^+(A) + \lambda_a^-(A)] \le \lambda_a(A),$$

$$\lambda_{ta}(tA) = \lambda_0(A-a), \quad \lambda_{ta}^+(tA) = \lambda_0^+(A-a), \quad \lambda_{ta}^-(tA) = \lambda_0^-(A-a),$$

$$\lambda_{-a}^+(-A) = \lambda_a^-(A), \quad \lambda_{-a}^-(-A) = \lambda_a^+(A),$$

$$\lambda_a(A \cup \{a\}) = \lambda_a(A), \quad \lambda_a^+(A \cup \{a\}) = \lambda_a^+(A), \quad \lambda_a^-(A \cup \{a\}) = \lambda_a^-(A).$$

Lemma 1. *Let $A \subset \mathbb{R}$, $a \in \mathbb{R}$ and $\alpha \in (0, 1)$ be fixed. If $\lambda_a^+(A) > \alpha$ then there exists a closed set $F \subset A \cup \{a\}$ such that $\lambda_a^+(F) > \alpha$.*

Proof. By Remark 1 we may assume that $a = 0$. Take an arbitrary β such that

$$\alpha < \beta < \lambda_a^+(A).$$

From the definition of $\lambda_0^+(A)$ we have

$$\lambda_0^+(A) = \liminf_{n\to\infty} \frac{m(A \cap [0, \frac{1}{n}])}{\frac{1}{n}} > \beta.$$

whence there exists n_0 such that

$$m\Big((A \cup \{0\}) \cap \Big[0, \frac{1}{n}\Big]\Big) > \frac{\beta}{n}, \quad n \in \mathbb{N},\ n \geq n_0.$$

As m is an inner measure, for any $n \geq n_0$, there is a closed set $F_n \subset (A \cup \{0\}) \cap [0, \frac{1}{n}]$ such that

$$m(F_n) > \frac{\beta}{n}, \quad n \in \mathbb{N},\ n \geq n_0.$$

Put

$$F := \{0\} \cup \bigcup_{n=n_0}^{\infty} F_n.$$

Obviously, $F \subset A \cup \{0\}$. Since $0 \in F$ and, for every $\varepsilon > 0$, there exists a positive integer $k = k(\varepsilon)$ such that the set

$$F \cap [\varepsilon, \infty] = [\varepsilon, \infty] \cap \bigcup_{n=n_0}^{k} F_n$$

is closed, it follows that F is a closed subset of $[0, \infty)$.

Now take an arbitrary $h \in (0, \frac{1}{n})$. There is a unique positive integer $p = p(h) \geq n_0$ such that

$$\frac{1}{p+1} \leq h < \frac{1}{p}.$$

Hence, as

$$m(F \cap [0, h]) \geq m\Big(F \cap \Big[0, \frac{1}{p+1}\Big]\Big) \geq m(F_{p+1}),$$

we have

$$\frac{m(F \cap [0, h])}{h} \geq \frac{m(F_{p+1})}{\frac{1}{p}} = \frac{p}{p+1} \frac{m(F_{p+1})}{\frac{1}{p+1}} > \frac{p}{p+1}\beta.$$

Since $\lim_{h \to 0} p(h)/(p(h)+1) = 1$, we hence get

$$\liminf_{h \to 0+} \frac{m(F \cap [0, h])}{h} \geq \beta$$

and, consequently, $\lambda_0^+(F) \geq \beta$. □

As an easy consequence of Lemma 1, we get the following

Remark 2. For any $A \subset \mathbb{R}$ and $a \in \mathbb{R}$,

$$\lambda_a^+(A) = \sup\{\lambda_a^+(F) : F \subset A \cup \{a\},\ F \text{ is closed}\}$$

(and similar formulas hold true for $\lambda_a^+(A)$ and $\lambda_a(A)$).

Lemma 2. *If $A_j \subset \mathbb{R}$, $a_j \in \mathbb{R}$ and $\alpha_j \in [0, 1]$ for $j \in \{1, \dots, n\}$ are such that*

$$\lambda_{a_j}^+(A_j) \geq 1 - \alpha_j, \quad j \in \{1, \dots, n\},$$

then

$$\lambda_{a_1+\cdots+a_n}^+(A_1 + \cdots + A_n) > 1 - \alpha_1 \cdots \alpha_n.$$

In particular, if $A \subset \mathbb{R}$, $a \in \mathbb{R}$ and $\alpha \in [0, 1]$ are such that

$$\lambda_a^+(A) \geq 1 - \alpha,$$

then

$$\lambda_{na}^+\Bigl(\sum_{j=1}^{n} A\Bigr) \geq 1 - \alpha^n.$$

Proof. We first show that this lemma holds true for $n = 2$. Take an arbitrary $\varepsilon > 0$ and, for the simplicity of notations, put

$$A := A_1,\ B := A_2; \quad a := a_1,\ b := a_2; \quad \alpha := \alpha_1 + \varepsilon,\ \beta := \alpha_2 + \varepsilon.$$

By the assumptions we have

$$\lambda_a^+(A) > 1 - \alpha, \quad \lambda_b^+(B) > 1 - \beta.$$

In view of Lemma 1 we may assume that $C := A \cup \{a\}$ and $D := B \cup \{b\}$ are closed. Moreover, by Remark 1, we may assume that $a = b = 0$. Thus we have

$$\lambda_0^+(C) > 1 - \alpha, \quad \lambda_0^+(D) > 1 - \beta.$$

It follows that there exists $\delta > 0$ such that, for every $h \in (0, \delta)$,

$$m\bigl((0, h) \setminus C\bigr) < \alpha h, \quad m\bigl((0, h) \setminus D\bigr) < \beta h.$$

Take $h < \delta$. To estimate $m((C + D) \cap [0, h])/h$, the average density of the set $C + D$ in the interval $[0, h]$, we shall consider two types of points of the set $C + D$.

(a) The points of the form $x + 0$ where $x \in C$ and, as D is closed, $0 \in D$. These points form the set C and, by the definition of δ, its measure is greater than $(1-\alpha)h$.

(b) Note that

$$(0,h) \setminus C = \bigcup_{k=1}^{\infty} (c_k, d_k),$$

where $(c_j, d_j) \cap (c_k, d_k) = \emptyset$ for $j \neq k$ and, as C is closed, all the points c_k, d_k are elements of C (for the obvious reason we have to admit that $c_k = d_k$ for some, or even for all $k \in \mathbb{N}$). We define the set E of points of the second type as

$$E := \bigcup_{k=1}^{\infty} \{c_k + y : y \in D \cap (0, d_k - c_k)\}.$$

Obviously $E \subset C + D$,

$$\begin{aligned} E &= \bigcup_{k=1}^{\infty} (c_k, d_k) \cap E = \bigcup_{k=1}^{\infty} \{c_k + y : y \in D \cap (0, d_k - c_k)\} \\ &= \bigcup_{k=1}^{\infty} (c_k + [D \cap (0, d_k - c_k)]) \end{aligned}$$

and

$$[(0,h) \setminus C] \setminus E = \bigcup_{k=1}^{\infty} \{c_k + [(0, d_k - c_k) \setminus [D \cap (0, d_k - c_k)]]\}.$$

Hence, as $d_k - c_k < \delta$ for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned} m([(0,h) \setminus C] \setminus E) &= \sum_{k=1}^{\infty} m((0, d_k - c_k) \setminus [D \cap (0, d_k - c_k)]) \\ &= \sum_{k=1}^{\infty} m((0, d_k - c_k) \setminus D) \le \sum_{k=1}^{\infty} (d_k - c_k)\beta \le \alpha\beta h. \end{aligned}$$

It follows that, for any $h \in (0, \delta)$,

$$m((0,h) \setminus (C + D)) \le m([(0,h) \setminus C] \setminus E) < \alpha\beta h.$$

We prove that the set $(C + D) \setminus (A + B)$ is of measure zero. Since $A + B \subset C + D$ and $(C + D) \setminus (A + B) \subset (A \cup B) \setminus (A + B)$, it is enough to show that the sets $A \setminus (A+B)$ and $B \setminus (A+B)$ are of measure zero. By the Lebesgue density theorem, it is enough to consider density points of the sets A and B.

Assume that $x \in A$ is a left-sided density point of A. Then, there is a number $h \in (0, \delta)$ such that $m((x-h, x) \cap A) > \beta h$. Since $m((0, h) \cap B) > (1-\beta)h$, so $x + [(-h, 0) \cap (-B)]$ is a subset of the interval $(x-h, x)$ with the measure greater than $(1-\beta)h$, and consequently, the set

$$E := [(x-h, x) \cap A] \cap \{x + [(-h, 0) \cap (-B)]\}$$

is nonempty. Thus, for arbitrary $t \in E$ we have $t \in A$ and $x - t \in B$, whence $x \in A + B$. It follows that $A \setminus (A+B)$ is of measure zero. A similar reasoning shows that $B \setminus (A+B)$ is of measure zero.

Now the previous step of the proof and the equality

$$(0, h) \setminus (A+B) = [(0, h) \setminus (C+D)] \cup \{(0, h) \cap [(C+D) \setminus (A+B)]\}$$

imply that

$$m\big((0, h) \setminus (A+B)\big) < \alpha\beta h.$$

By the definition of the right-lower density of $A+B$ at the point 0, we hence get

$$\lambda_0^+(A+B) \geq 1 - \alpha\beta,$$

that is

$$\lambda_0^+(A_1 + A_2) \geq 1 - (\alpha_1 + \varepsilon)(\alpha_2 + \varepsilon),$$

whence, letting $\varepsilon \to 0$, we obtain

$$\lambda_0^+(A_1 + A_2) \geq 1 - \alpha_1\alpha_2.$$

Thus we have shown that

$$\lambda_{a_1+a_2}^+(A_1 + A_2) \geq 1 - \alpha_1\alpha_2,$$

which proves that the lemma holds true for $n = 2$.

Now an obvious inductive argument completes the proof. □

Taking

$$\alpha_j := 1 - \lambda_{a_j}^+(A_j), \quad j \in \{1, \dots, n\}$$

in Lemma 2 gives the following

Corollary 1. *If $A_j \subset \mathbb{R}$ and $a_j \in \mathbb{R}$ for $j \in \{1, \dots, n\}$, then*

$$\lambda^+_{a_1+\dots+a_n}\left(\sum_{j=1}^{n} A_j\right) \geq 1 - \left(1 - \lambda^+_{a_1}(A_1)\right) \cdot \ldots \cdot \left(1 - \lambda^+_{a_{n1}}(A_n)\right).$$

In particular, for all $A \subset \mathbb{R}$, $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\lambda^+_{na}\left(\sum_{j=1}^{n} A\right) \geq 1 - \left(1 - \lambda^+_a(A)\right)^n.$$

Now we prove the following

Lemma 3. *Suppose that $A, B \subset \mathbb{R}$, $a, b \in \mathbb{R}$, $h > 0$ and*

$$m(A \cap (a, a+h)) = \alpha h, \quad m(B \cap (b, b+h)) = \beta h$$

for some $\alpha, \beta > 0$. If $\alpha + \beta > 1$ then there exist $x \in A \cap (a, a+h)$ and $y \in B \cap (b, b+h)$ such that

$$x + y = a + b + h.$$

Proof. Without any loss of generality we can assume that $a = b = 0$. Put $A_h := A \cap (0, h)$, $B_h := B \cap (0, h)$ and note that

$$m(h - B_h) = m(B_h) = \beta h.$$

Suppose that the lemma is false. Then

$$A_h \cap (h - B_h) = \emptyset$$

and, since $A_h, B_h \subset (0, h)$, we would have

$$\alpha h + \beta h = m(A_h) + m(h - B) \leq m((0, h)) = h,$$

and, consequently, $\alpha + \beta \leq 1$. This contradiction completes the proof. □

This lemma implies the following:

Corollary 2. *If $A, B \subset \mathbb{R}$, $a, b \in \mathbb{R}$ and*

$$\lambda^+_a(A) + \lambda^+_b(B) > 1,$$

then there exists $\delta > 0$ such that

$$(a + b) + (0, \delta) \subset A + B.$$

This corollary is a special case of the following

Theorem 1. *If $A_j \subset \mathbb{R}$, $a_j \in \mathbb{R}$ for $j \in \{1, \dots, n\}$ and, for some $k \in \{1, \dots, n\}$,*

$$\prod_{j=1}^{k}\left(1-\lambda_{a_j}^{+}(A_j)\right)+\prod_{j=k+1}^{n}\left(1-\lambda_{a_j}^{+}(A_j)\right)<1,$$

then there exists $\delta > 0$ such that

$$(a_1+\cdots+a_n)+(0,\delta)\subset A_1+\cdots+A_n.$$

Proof. Put

$$A := A_1+\cdots+A_k, \quad B := A_{k+1}+\cdots+A_n;$$
$$a := a_1+\cdots+a_k, \quad b := a_{k+1}+\cdots+a_n.$$

Applying Corollary 1 and the assumption we obtain

$$\begin{aligned}
&\lambda_a^{+}(A)+\lambda_b^{+}(B)\\
&=\lambda_{a_1+\cdots+a_k}^{+}(A_1+\cdots+A_k)+\lambda_{a_{k+1}+\cdots+a_n}^{+}(A_{k+1}+\cdots+A_n)\\
&\geq 1-\prod_{j=1}^{k}\left(1-\lambda_{a_j}^{+}(A_j)\right)+1-\prod_{j=k+1}^{n}\left(1-\lambda_{a_j}^{+}(A_j)\right)>1.
\end{aligned}$$

Now the result follows from Corollary 2. □

Corollary 3. *If $A \subset \mathbb{R}$ and $a \in \mathbb{R}$ are such that $\lambda_a^{+}(A) > 0$, then there exist $n \in \mathbb{N}$ and $\delta > 0$ such that*

$$na+(0,\delta)\subset\sum_{j=1}^{n}A.$$

Proof. Let $A_j := A$ and $a_j := a$ for all $j \in \mathbb{N}$. Take $k := 1$ and put $t := \lambda_a^{+}(A)$. As $0 < t \leq 1$, the number

$$\begin{aligned}
&\prod_{j=1}^{k}\left(1-\lambda_{a_j}^{+}(A_j)\right)+\prod_{j=k+1}^{n}\left(1-\lambda_{a_j}^{+}(A_j)\right)\\
&\quad=\left(1-\lambda_a^{+}(A)\right)^{k}+\left(1-\lambda_a^{+}(A)\right)^{n-k}=(1-t)+(1-t)^{n-1}
\end{aligned}$$

is less than 1 for all sufficiently large $n \in \mathbb{N}$. The result follows from Theorem 1. □

In this context the following problem arises. Let $n \in \mathbb{N}$, $n \geq 2$, be fixed. Find the infimum of all values κ such that for all $A \subset \mathbb{R}$ and $a \in \mathbb{R}$,

$$\lambda_a^+(A) > \kappa \quad \text{implies that} \quad na + (0, \delta) \subset \sum_{j=1}^{n} A \text{ for some } \delta > 0.$$

Relying on the proof of the above corollary, we get the following estimation.

Remark 3. Let $A \subset \mathbb{R}$ and $a \in \mathbb{R}$. Denote by d_n the smallest positive root of the polynomial

$$p_n(x) := 1 - x + (1-x)^{n-1} - 1.$$

As for any $n \in \mathbb{N}$, $n \geq 2$, we have $p_n(0) = 1$, $p_n(1) = -1$, the number d_n is well defined, $0 < d_n < 1$ and $\lim_{n\to\infty} d_n = 0$. It is easy to verify the following fact:

If $\lambda_a^+(A) > d_n$ then there is $\delta > 0$ such that

$$na + (0, \delta) \subset \sum_{j=1}^{n} A.$$

Moreover

$$d_2 = \frac{1}{2};$$

$$d_3 = \frac{3-\sqrt{5}}{2}, \quad \frac{1}{3} < d_3 < \frac{2}{5};$$

$$d_4 = 1 + \left(\frac{\sqrt{93}}{18} - \frac{1}{2}\right)^{1/3} - \left(\frac{\sqrt{93}}{18} + \frac{1}{2}\right)^{1/3}, \quad \frac{1}{4} < d_4 < \frac{1}{3}.$$

Remark 4. In the above remark we applied Theorem 1 for $k = 1$. This procedure can be extended for other $k \in \mathbb{N}$.

To see this, let us fix $n \in \mathbb{N}$. Taking $A_j := A$ and $a_j := a \in \mathbb{R}$ for all $j \in \{1, \ldots, n\}$ and $t := \lambda_a^+(A)$, the main assumption of Theorem 1 can be written in the following form:

There exists $k \in \{1, \ldots, n\}$ such that

$$(1-t)^k + (1-t)^{n-k} - 1 < 0.$$

Since for each $k \in \{1, \ldots, n\}$, the polynomial $p_{n,k}$ defined by

$$p_{n,k}(x) := (1-x)^k + (1-x)^{n-k} - 1, \quad x \in \mathbb{R},$$

is decreasing in $[0, 1]$ and

$$p_{n,k}(0) = 1, \quad p_{n,k}(1) = -1,$$

this polynomial has a unique zero $r_{n,k} \in (0,1)$. Put

$$r_n := \min\{r_{n,k} : k = 1, \dots, n\}.$$

Now it is easy to see that the set $A \subset \mathbb{R}$ and $a \in \mathbb{R}$ satisfy the assumptions of Theorem 1 if, for some $k \in \{1, \dots, n\}$,

$$r_{n,k} < t \quad \text{where } t = \lambda_a^+(A),$$

or, equivalently, if

$$r_n < \lambda_a^+(A).$$

Since $p_{n,k} = p_{n,n-k}$ and, for all $k \le \frac{n-1}{2}$, $k \in \mathbb{N}$, and $x \in [0,1]$ we have

$$p_{n,k}(x) - p_{n,k+1}(x) = x(1-x)^k\big[1 - (1-x)^{n-2k-1}\big] \ge 0,$$

the decreasing monotonicity of $p_{n,k}$ in $[0,1]$ implies that

$$r_n = \begin{cases} r_{2k,k} & \text{if } n = 2k, \\ r_{2k+1,k} & \text{if } n = 2k+1. \end{cases}$$

(Note also that here $r_{2k+1,k} = r_{2k+1,k+1}$.)

First consider the case when $n = 2k$. Then

$$p_{n,k}(x) = 2(1-x)^k - 1, \quad x \in \mathbb{R},$$

and, consequently,

$$r_{2k} = r_{2k,k} = 1 - 2^{-1/k}.$$

Note that

$$r_2 = \frac{1}{2}; \quad \frac{1}{2k} < r_{2k} < \frac{\ln 2}{k} \quad \text{for } k \in \mathbb{N},\ k > 1. \tag{1}$$

To show this, observe first that, by the mean-value theorem, for any $x > 0$, there is $c = c(x) \in (x, x+1)$ such that

$$(x+1)2^{-1/(x+1)} - x2^{-1/x} = 2^{-1/c}\Big(1 + \frac{\ln 2}{c}\Big).$$

Since $(1 + (\ln 2)x) < 2^x$ for all $x > 0$, it follows that

$$(x+1)2^{-1/(x+1)} - x2^{-1/x} < 1, \quad x > 0.$$

Setting $x = k$ in this inequality we conclude that

$$kr_{2k} < (k+1)r_{2(k+1)}, \quad k \in \mathbb{N}.$$

i.e., the sequence $(kr_{2k})_{k=1}^{\infty}$ is strictly increasing. Since

$$\lim_{k\to\infty} kr_{2k} = \ln 2,$$

we hence get

$$r_{2k} < \frac{\ln 2}{k}, \quad k \in \mathbb{N}.$$

It is easy to check that the inequality

$$\frac{1}{2k} < r_{2k}, \quad k \in \mathbb{N},\ k \geq 2,$$

is true for $k = 2$. Suppose it holds true for some $k \in \mathbb{N}$, $k \geq 2$. Hence, by the increasing monotonicity of the sequence $(kr_{2k})_{k=1}^{\infty}$ we get

$$\frac{1}{2(k+1)} = \frac{k}{k+1}\frac{1}{2k} < \frac{k}{k+1}r_{2k} = \frac{1}{k+1}kr_{2k}$$
$$< \frac{1}{k+1}(k+1)r_{2(k+1)} = r_{2(k+1)}$$

and the induction completes the proof of (1).

Now assume that $n = 2k + 1$. In this case we have

$$p_{2k+1,k}(x) = (2-x)(1-x)^k - 1, \quad x \in \mathbb{R}.$$

We shall show that

$$p_{2k+1,k}\left(\frac{1}{2k+1}\right) > 0, \quad k \in \mathbb{N}, \tag{2}$$

i.e., that

$$\left(2 - \frac{1}{2k+1}\right)\left(1 - \frac{1}{2k+1}\right)^k > 1, \quad k \in \mathbb{N},$$

which can be written in the form

$$\left(1 + \frac{1}{2k}\right)^{2k} < \left(\frac{4k+1}{2k+1}\right)^2, \quad k \in \mathbb{N}.$$

Of course we have

$$\left(1 + \frac{1}{2k}\right)^{2k} < e, \quad k \in \mathbb{N}.$$

Since the sequence $a_k := (\frac{4k+1}{2k+1})^2$ is increasing and $a_2 > e$, we conclude that inequality (2) holds true for all $k \in \mathbb{N}$, $k \geq 2$. It is easy to check that this inequality holds also for $k = 1$.

Inequality (2) implies that

$$\frac{1}{2k+1} < r_{2k+1}.$$

Since $p_{n,k}(x) \geq p_{n,k+1}(x)$, we get

$$r_{2k+1} < r_{2k} < \frac{2\ln 2}{k}.$$

Thus we have proved that

$$\frac{1}{2k+1} < r_{2k+1} < \frac{2\ln 2}{k}, \quad k \in \mathbb{N}.$$

Hence, applying Theorem 1, we obtain the following

Corollary 4. *Let $A \subset \mathbb{R}$, $a \in \mathbb{R}$, $n \in \mathbb{N}$, $n > 1$. If $\lambda_a^+(A) > r_n$ then there is $\delta > 0$ such that*

$$2na + (0, \delta) \subset \sum_{j=1}^{2n} A.$$

Moreover, if $n = 2k$ then

$$r_n = 1 - 2^{-1/k};$$

if $n = 2k+1$ then

$$\frac{1}{n} < r_n < \frac{2\ln 2}{n-1}.$$

Remark 5. Of course, Theorem 1, the earlier lemmas and corollaries remain true if we replace "+" by "−" or if we omit "+".

Theorem 2. *If $A_j \subset \mathbb{R}$, $a_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$ and, for some $k, l \in \{1, \ldots, n\}$,*

$$\prod_{j=1}^{k}\left(1 - \lambda_{a_j}^+(A_j)\right) + \prod_{j=k+1}^{n}\left(1 - \lambda_{a_j}^+(A_j)\right) < 1,$$

$$\prod_{j=1}^{l}\left(1 - \lambda_{a_j}^-(A_j)\right) + \prod_{j=l+1}^{n}\left(1 - \lambda_{a_j}^-(A_j)\right) < 1,$$

then there exists $\delta > 0$ such that

$$(a_1 + \cdots + a_n) + (-\delta, \delta) \subset A_1 + \cdots + A_n,$$

that is $a_1 + \cdots + a_n$ is an inner point of the set $A_1 + \cdots + A_n$.

Remark 6. To see that the theorem of Steinhaus is a consequence of Theorem 2, take two sets $A_1, A_2 \subset \mathbb{R}$ of positive Lebesgue measure. In view of the Lebesgue density theorem, there are $a_1 \in A_1, a_2 \in A_2$ of the density one. Consequently,

$$\lambda_{a_1}^-(A_1) = \lambda_{a_2}^-(A_2) = 1 = \lambda_{a_1}^+(A_1) = \lambda_{a_2}^+(A_2).$$

Since each of the assumed inequalities of Theorem 2 reduces to the trivial inequality $0 < 1$, the result holds true.

3 Raikov's theorem and its consequence

Given $A \subset \mathbb{R}, a \in \mathbb{R}$ and $x > 0$. The number

$$\pi_{(a,a+x)}(A) := \inf\left\{\frac{m(A \cap (a, a+h))}{h} : h \in (0, x)\right\}$$

is called *the density of the set A on the interval $(a, a+x)$*.

The theorem of Raikov [11] reads as follows:

Theorem 3. *For all $A, B \in \mathbb{R}$; $a, b \in \mathbb{R}$ and $x > 0$,*

$$\pi_{(a+b,a+b+x)}(A+B) \geq \min\{\pi_{(a,a+x)}(A) + \pi_{(b,b+x)}(B), 1\}.$$

If moreover,

$$\pi_{(a,a+x)}(A) + \pi_{(b,b+x)}(B) > 1,$$

then the set $A + B$ covers the interval $(a+b, a+b+x)$.

Note that this result is formulated for two sets and it does not use the density of a set at a point. However, since

$$\lambda_a^+(A) = \lim_{x \to 0+} \pi_{(a,a+x)}(A),$$

we hence get

$$\lambda_{a+b}^+(A+B) \geq \min\{\lambda_a^+(A) + \lambda_b^+(B), 1\}.$$

If moreover,

$$\lambda_a^+(A) + \lambda_b^+(B) > 1,$$

then there exists $\delta > 0$ such that $\pi_{(a,a+\delta)}(A) + \pi_{(b,b+\delta)}(B) > 1$. By Raikov's theorem, the set $A + B$ covers the interval $(a+b, a+b+\delta)$.

Therefore, by an easy induction, we obtain the following

Theorem 4. *Let $A_j \subset \mathbb{R}$ and $a_j \in \mathbb{R}$ for $j \in \{1, \dots, n\}$. If*

$$\sum_{j=1}^{n} \lambda_{a_j}^{+}(A_j) > 1,$$

then there is $\delta > 0$ such that

$$(a_1 + \dots + a_n) + (0, \delta) \subset A_1 + \dots + A_n.$$

In particular, if $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, $\lambda_a^+(A) > 0$ and $n \in \mathbb{N}$ is such that $n\lambda_a^+(A) > 1$, then there is $\delta > 0$ such that

$$na + (0, \delta) \subset \sum_{j=1}^{n} A.$$

(The result remains true when replacing λ^+ by λ^- or λ.)

Remark 7. Let $n = 2$, $A_1 = A_2 := A$ and $a_1 = a_2 := 0$. According to Theorem 3, if $\lambda_a^+(A) > \frac{1}{2}$, then $(0, \delta) \subset A + A$ for some $\delta > 0$. By Remark 3, as $d_2 = \frac{1}{2}$, the same implication gives Theorem 1. Consequently, in this case the results by Świątkowski and Raikov coincide.

For $n \geq 3$, $A_1 = \dots = A_n := A$ and $a_1 = \dots = a_n = 0$, according to Theorem 4, if $\lambda_a^+(A) > \frac{1}{n}$ then $(0, \delta) \subset \sum_{j=1}^{n} A$ for some $\delta > 0$. By Remarks 3 and 4 we have $d_n > \frac{1}{n}$ for $n \geq 3$. In this case Theorem 4 is a little better than Theorem 1.

4 An application

We begin this section with the following

Theorem 5. *Let $\beta, \alpha > 0$ and a set $A \subset (0, \infty)$ such that $\lambda_0^+(A) > 0$ be arbitrarily fixed. Suppose that $f : (0, \infty) \to \mathbb{R}$ satisfies the inequality*

$$f(x + \alpha y) \leq f(x) + \beta f(y), \quad x, y > 0. \tag{3}$$

If the function f satisfies the condition

$$\limsup_{x \to 0+} f\big|_A(x) \leq 0, \tag{4}$$

then, for every $x > 0$, the one-sided limits

$$f_-(x) := \lim_{y \to x-} f(y), \quad f_+(x) := \lim_{y \to x+} f(y)$$

exist and satisfy the inequality

$$f_+(x) \leq f(x) \leq f_-(x).$$

Proof. From (3) we have

$$f\Big(\sum_{j=1}^{n} x_j\Big) \le \sum_{j=1}^{n} \beta^{j-1} f\Big(\frac{x_j}{\alpha^{j-1}}\Big), \quad n \in \mathbb{N},\ x_1, \dots, x_n > 0. \tag{5}$$

Indeed, assuming that this inequality holds true for some $n \in \mathbb{N}$ and applying (3), we hence get

$$\begin{aligned} f\Big(\sum_{j-1}^{n+1} x_j\Big) &= f\Big(x_1 + \alpha \sum_{j=2}^{n+1} \frac{x_j}{\alpha}\Big) \\ &\le f(x_1) + \beta f\Big(\sum_{j=2}^{n+1} \frac{x_j}{\alpha}\Big) \\ &= f(x_1) + \beta f\Big(\sum_{j=1}^{n} \frac{x_{j+1}}{\alpha}\Big) \\ &\le f(x_1) + \beta\Bigg[\sum_{j=1}^{n} \beta^{j-1} f\bigg(\frac{\frac{x_{j+1}}{\alpha}}{\alpha^{j-1}}\bigg)\Bigg] \\ &= f(x_1) + \sum_{j=1}^{n} \beta^{j} f\Big(\frac{x_{j+1}}{\alpha^{j}}\Big) = \sum_{j=1}^{n+1} \beta^{j-1} f\Big(\frac{x_j}{\alpha^{j-1}}\Big) \end{aligned}$$

for all $x_1, \dots, x_{n+1} > 0$, and the induction completes the proof of (5).

Put $A_j := \alpha^{j-1} A$ for $j \in \mathbb{N}$. By Remark 1 we have $\lambda_0^+(A_j) = \lambda_0^+(A)$ for all $j \in \mathbb{N}$. By Theorem 1 or Theorem 4, there exist $n \in \mathbb{N}$ and $\delta > 0$ such that

$$(0, \delta) \subset \sum_{j=1}^{n} \alpha^{j-1} A.$$

Take an arbitrary sequence $x_k \in (0, \delta)$ such that $\lim_{k\to\infty} x_k = 0$. Hence there exists $y_{k,j} \in \alpha^{j-1} A$ for $j = 1, \dots, n$ such that

$$x_k = \sum_{j=1}^{n} y_{k,j}, \quad k \in \mathbb{N}.$$

From (5) we have

$$f(x_k) = f\Big(\sum_{j=1}^{n} y_{k,j}\Big) \le \sum_{j=1}^{n} \beta^{j-1} f\Big(\frac{y_{k,j}}{\alpha^{j-1}}\Big), \quad k \in \mathbb{N}.$$

Since

$$\frac{y_{k,j}}{\alpha^{j-1}} \in A, \quad k \in \mathbb{N},\ j = 1, \ldots, n,$$

making use of (4), we obtain

$$\limsup_{k\to\infty} f(x_k) \le \sum_{j=1}^{n} \beta^{j-1} \limsup_{k\to\infty} f\left(\frac{y_{k,j}}{\alpha^{j-1}}\right) \le 0,$$

whence, as $x_k \in (0, \delta)$ such that $\lim_{k\to\infty} x_k = 0$ is arbitrary, we have

$$\limsup_{x\to 0+} f(x) \le 0. \tag{6}$$

Inequality (3) and condition (4) easily imply that f is bounded from above on any interval $(0, c)$ for $c > 0$. Let us fix an arbitrary $c > 0$ and take $M > 0$ such that $f(x) < M$ for all $x \in (0, c)$. We shall show that f is bounded from below in some right vicinity of 0. On the contrary, assume that there is a sequence $y_k > 0$, $\lim_{k\to\infty} y_k = 0$ such that $\lim_{k\to\infty} f(y_k) = -\infty$. Then, by (3), for any $x \in (0, c)$ and sufficiently large $k \in \mathbb{N}$, we would have

$$f(x) = f\left((x - \alpha y_k) + \alpha y_k\right) \le f(x - \alpha y_k) + \beta f(y_k) \le M + \beta f(y_k),$$

whence, letting $k \to \infty$, we obtain $f(x) = -\infty$. This is a contradiction, as f is real-valued.

The boundedness from below of the function f implies that $\liminf_{x\to 0+} f(x)$ is finite. From (3) we have

$$\liminf_{x\to 0+} f(x) = \liminf_{\substack{x\to 0+\\ y\to 0+}} f(x + \alpha y) \le \liminf_{x\to 0+} f(x) + \beta \liminf_{y\to 0+} f(y),$$

whence, as $\beta > 0$,

$$\liminf_{x\to 0+} f(x) \ge 0.$$

This inequality and (6) imply that

$$\lim_{x\to 0+} f(x) = 0.$$

Now take an arbitrary $x > 0$ and two sequences u_k, v_k satisfying the inequalities $x < u_k < v_k$ with $\lim_{k\to\infty} v_k = x$, and such that

$$\liminf_{u\to x+} f(u) = \lim_{k\to\infty} f(u_k), \quad \limsup_{u\to x+} f(u) = \lim_{k\to\infty} f(v_k).$$

From (3) we have

$$f(v_k) = f\left(u_k + \alpha\frac{v_k - u_k}{\alpha}\right) \le f(u_k) + \beta f\left(\frac{v_k - u_k}{\alpha}\right), \quad k \in \mathbb{N},$$

whence, letting $k \to \infty$, we obtain

$$\limsup_{u \to x+} f(u) \le \liminf_{u \to x+} f(u).$$

It follows that $f_+(x)$ exists. We omit a similar argument showing that $f_-(x)$ exists.

Now, taking a sequence u_k such that $u_k > x$, $\lim_{k\to\infty} u_k = x$, in view of (3),

$$f(u_k) = f\left(x + \alpha\frac{u_k - x}{\alpha}\right) \le f(x) + \beta f\left(\frac{u_k - x}{\alpha}\right), \quad k \in \mathbb{N},$$

whence, letting $k \to \infty$, we obtain $f(x+) \le f(x)$.

The proof of the inequality $f(x) \le f(x-)$ is analogous. □

Remark 8. Taking $\alpha = \beta = 1$ in the above result we obtain an improvement of a classical result for subadditive functions [2, p. 248, Theorem 7.8.3], as well as an earlier result of the present author (cf. [10, p. 31]).

Now we can prove the following converse theorem for Minkowski's inequality:

Theorem 6. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is one-to-one, onto and there is a set $C \subset [0, \infty)$ such that $\lambda_0^+(C) > 0$ and the restriction $\varphi^{-1}|_C$ is right continuous at 0, and

$$\varphi^{-1}\left(\int_\Omega \varphi \circ (x + y) d\mu\right) \le \varphi^{-1}\left(\int_\Omega \varphi \circ x d\mu\right) + \varphi^{-1}\left(\int_\Omega \varphi \circ y d\mu\right) \quad (7)$$

for all nonnegative μ-integrable step functions $x, y : \Omega \to \mathbb{R}$. Then there exists $p \ge 1$ such that

$$\varphi(t) = \varphi(1)t^p, \quad t \ge 0.$$

Proof. Taking

$$x = \frac{\varphi^{-1}(s)}{\mu(A)}\chi_A, \quad y = \frac{\varphi^{-1}(t)}{\mu(B \setminus A)}\chi_{B\setminus A}$$

in inequality (7) we get

$$\varphi^{-1}(s+t) \leq \varphi^{-1}(s) + \varphi^{-1}(t), \quad s, t \geq 0,$$

that is, φ^{-1} is subadditive in $[0, \infty)$. Since φ^{-1} is onto, there is $t \geq 0$ such that $\varphi^{-1}(t) = 0$. From the subadditivity of φ^{-1} we have $\varphi^{-1}(2t) \leq 2\varphi^{-1}(t) = 0$, whence $\varphi^{-1}(2t) = 0$. The injectivity of φ^{-1} implies that $2t = t$, whence $t = 0$ and, consequently, $\varphi^{-1}(0) = 0$. Taking into account the nonnegativity of φ, the right continuity of $\varphi^{-1}|_A$ at 0, and applying Theorem 5 with $\alpha = \beta = 1$ and $f := \varphi^{-1}$, we conclude that φ^{-1} is continuous at 0. Now the result follows from [4, Theorem 1]. □

Remark 9. Theorem 6 improves the main result of [4] where it is assumed that φ^{-1} is right continuous at 0 and $\varphi(0) = 0$.

Let us also note the following easy to prove

Proposition 1. *Let $p \in \mathbb{N}$ and $\alpha_j, \beta_j \in (0, \infty)$ for $j \in \{1, \ldots, p\}$ be fixed. Suppose that $A_1, \ldots, A_p \subset (0, \infty)$ are such that $\lambda_0^+(A_i) > 0$ and put*

$$\mathbf{A}_i := \{(0, \ldots, 0, x_i, 0, \ldots, 0) : x_i \in A_i\}, \quad i = 1, \ldots, p.$$

If a function $f : ([0, \infty)^p \setminus \{\mathbf{0}\}) \to \mathbb{R}$ satisfies the inequality

$$f\left(\sum_{j=1}^{p} \alpha_j \mathbf{x}_j\right) \leq \sum_{j=1}^{p} \beta_j f(\mathbf{x}_j), \quad \mathbf{x}_1, \ldots, \mathbf{x}_p \in [0, \infty)^p \setminus \{\mathbf{0}\}$$

and

$$\limsup_{\mathbf{x} \to \mathbf{0}+} f\big|_{\mathbf{A}_i}(\mathbf{x}) \leq 0,$$

then

$$\limsup_{\mathbf{x} \to \mathbf{0}+} f(\mathbf{x}) \leq 0.$$

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Bibliography

[1] F. Bernstein and G. Doetsch, Zur Theorie der konvexen Funktionen, *Math. Ann.* **76** (1915), 514–526.

[2] E. Hille and S. P. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ. **31**, American Mathematical Society, Providence, 1957.

[3] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, PWN, Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985.

[4] J. Matkowski, The converse of the Minkowski inequality theorem and its generalization, *Proc. Amer. Math. Soc.* **109** (1990), 663–675.

[5] J. Matkowski and T. Świątkowski, Quasi-monotonicity, subadditive bijections of $\mathbb{R}_+$, and a characterization of L^p-norm, *J. Math. Anal. Appl.* **154** (1991), 493–506.

[6] J. Matkowski and T. Świątkowski, On subadditive functions, *Proc. Amer. Math. Soc.* **119** (1993), 187–197.

[7] J. Matkowski and T. Świątkowski, Subadditive functions and partial converses of Minkowski's and Mulholland's inequalities, *Fund. Math.* **143** (1993), 75–85.

[8] A. Ostrowski, Zur Theorie der konvexen Funktionen, *Comment. Math. Helv.* **1** (1929), 157–159.

[9] J. C. Oxtoby, *Measure and Category*, Springer, Berlin, 1980.

[10] M. Pycia, Linear functional inequalities – a general theory and new special cases, *Dissertationes Math.* **438** (2006), 1–62.

[11] D. A. Raikov, On the addition of point sets in the sense of Schnirelmann (in Russian), *Math. Sbornik* **5** (1939), 425–440.

[12] W. Sierpiński, Sur les fonctions convexes mesurables, *Fund. Math.* **1** (1920), 116–122.

[13] H. Steinhaus, Sur les distances des points des ensembles de mesure positive, *Fund. Math.* **1** (1919), 274–297.

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