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Fixed points and iterations of mean-type mappings

Research Article

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Abstract: Let (X, d) be a metric space and $T: X \to X$ a continuous map. If the sequence $(T^n)_{n\in\mathbb{N}}$ of iterates of T is pointwise convergent in X, then for any $x \in X$, the limit

$$\mu_T(x) = \lim_{n\to\infty} T^n(x)$$

is a fixed point of T. The problem of determining the form of μ_T leads to the invariance equation $\mu_T \circ T = \mu_T$, which is difficult to solve in general if the set of fixed points of T is not a singleton. We consider this problem assuming that $X = I^p$, where I is a real interval, $p \geq 2$ a fixed positive integer and T is the mean-type mapping $\mathbf{M} = (M_1, \ldots, M_p)$ of I^p. In this paper we give the explicit forms of $\mu_{\mathbf{M}}$ for some classes of mean-type mappings. In particular, the classical Pythagorean harmony proportion can be interpreted as an important invariance equality. Some applications are presented. We show that, in general, the mean-type mappings are not non-expansive.

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Keywords: Mean • Homogeneous mean • Mean-type mapping • Invariant mean • Iteration • Fixed point

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Introduction

Let (X, d) be a nonempty metric space, $T: X \to X$ a continuous mapping and let $\mathrm{Fix}(T)$ denote the set of fixed points of T. Assume that the sequence $(T^n)_{n\in\mathbb{N}}$ of iterates of T is pointwise convergent in X, i.e. that, for any $x \in X$, the limit

$$\mu_T(x) = \lim_{n\to\infty} T^n(x)$$

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exists and, clearly, μ_T maps X onto $\mathrm{Fix}(T)$. Since

$$\mu_T(x) = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T^n(Tx) = \mu_T(T(x)), \qquad x \in X,$$

the mapping μ_T is *invariant with respect to* T, that is μ_T satisfies the equality

$$\mu_T \circ T = \mu_T.$$

In general, if for instance $\mathrm{Fix}(T)$ is not a singleton, the problem of determining the explicit form of μ_T can be nontrivial.

The *mean-type mappings* form a broad class of maps for which this problem seems to be especially interesting. For an interval $I \subseteq \mathbb{R}$ and a fixed positive integer $p \geq 2$, put $X = I^p$. A map $\mathbf{M}\colon I^p \to I^p$ is called a *mean-type mapping* if $\mathbf{M} = (M_1, \ldots, M_p)$ where the coordinate functions $M_1, \ldots, M_p\colon I^p \to I$ are means, Section 1. For any mean-type mapping $\mathbf{M}$, the set $\mathrm{Fix}(\mathbf{M})$ is the diagonal of I^p and, under some general weak conditions, the sequence of iterates $(\mathbf{M}^n)_{n\in\mathbb{N}}$ is pointwise convergent to the *invariant mean-type mapping* of the same invariant coordinate mean, cf. Theorem 1.7.

In Section 2, for some classes of mean-type mappings the explicit forms of their invariant means are presented. In particular, applying Proposition 2.1 with $p = 2$ we conclude that the sequence of iterates of the mean-type mapping $\mathbf{M}\colon (0,\infty)^2 \to (0,\infty)^2$, $\mathbf{M} = (A, H)$, where $A(x, y) = (x + y)/2$ is the arithmetic mean and $H(x, y) = 2xy/(x + y)$ is the harmonic mean, converges to the mean-type mapping (G, G), where $G(x, y) = \sqrt{xy}$ for $x, y > 0$. That is,

$$\lim_{n\to\infty} (A, H)^n = (G, G).$$

This is a consequence of the equality $G \circ (A, H) = G$, the invariance of the geometric mean G with respect to the mean-type mapping (A, H). It is interesting that this invariance equality is just the Pythagorean harmony proportion of the means A, G and H. In Section 3 we apply the invariant means in solving some functional equations. It is also observed in Section 4 that, in general, the mean-type mappings are not non-expansive. In particular the relation $\mathrm{Fix}(\mathbf{M}) \neq \emptyset$, as well as the pointwise convergence of the sequence of iterates $(\mathbf{M}^n)_{n\in\mathbb{N}_0}$, cannot be deduced from the respective part of the fixed point theory.

1. Mean-type mappings, iterations, invariant means

Let $I \subset \mathbb{R}$ be an interval and fix $p \in \mathbb{N}$, $p \geq 2$. A function $M\colon I^p \to \mathbb{R}$ is said to be a *p-variable mean* on I (briefly, a *mean* on I) if, for all $x_1, \ldots, x_p \in I$,

$$\min(x_1, \ldots, x_p) \leq M(x_1, \ldots, x_p) \leq \max(x_1, \ldots, x_p).$$

A mean M in I is called *strict* if these inequalities are sharp whenever

$$\min(x_1, \ldots, x_p) < \max(x_1, \ldots, x_p).$$

Remark 1.1.
If $M\colon I^p \to \mathbb{R}$ is a mean then M maps I^p onto I and, moreover, M is reflexive. That is, for all $x \in I$,

$$M(x, \ldots, x) = x.$$

Let us also note

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Remark 1.2.
If a function $M\colon I^p \to \mathbb{R}$ is reflexive and (strictly) increasing with respect to each variable, then M is a (strict) mean on I.

The following is easy to verify.

Remark 1.3.
Let $M\colon I^p \to \mathbb{R}$ be an arbitrary function. Then the following conditions are equivalent:

(i) M is a mean;

(ii) $M(J^p) \subset J$ for every subinterval $J \subset I$;

(iii) $M(J^p) = J$ for every subinterval $J \subset I$.

A mapping $\mathbf{M}\colon I^p \to I^p$ is referred to as *mean-type* if there are some means $M_i\colon I^p \to I$, $i = 1, \ldots, p$, such that $\mathbf{M} = (M_1, \ldots, M_p)$. Note the following obvious

Remark 1.4.
Let $\mathbf{M}\colon I^p \to I^p$ be a mean-type mapping. Then $(x_1, \ldots, x_p) \in I^p$ is a fixed-point of $\mathbf{M}$ if and only if $x_1 = \ldots = x_p$. Thus $\operatorname{Fix}(\mathbf{M}) = \Delta(I^p)$, where

$$\Delta(I^p) = \{(x_1, \ldots, x_p) \in I^p : x_1 = \ldots = x_p\}$$

denotes the diagonal of I^p.

We say that the mean-type mapping $\mathbf{M} = (M_1, \ldots, M_p)$ is *strict* if each of its coordinate means $M_1, \ldots, M_p$ is a strict mean. In the case when $I = (0, \infty)$, we say that the mean-type mapping $\mathbf{M}$ is *positively homogeneous* if each of its coordinate means $M_1, \ldots, M_p$ is positively homogeneous, i.e., if

$$M_i(tx_1, \ldots, tx_p) = tM_i(x_1, \ldots, x_p), \qquad t, x_1, \ldots, x_p > 0, \quad i = 1, \ldots, p.$$

Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $\mathbf{M}\colon I^p \to I^p$ is a mean-type mapping then, clearly, the sequence $(\mathbf{M}^n)_{n=0}^{\infty}$ of the iterates of $\mathbf{M}$,

$$\mathbf{M}^0 = \mathrm{Id}|_{I^p}, \qquad \mathbf{M}^{n+1} = \mathbf{M} \circ \mathbf{M}^n \qquad \text{for} \quad n \in \mathbb{N}_0,$$

is well defined. The following is obvious.

Remark 1.5.
Suppose that $\mathbf{M}\colon I^p \to I^p$, $\mathbf{M} = (M_1, \ldots, M_p)$, is a mean-type mapping of I^p. Then, for each $n \in \mathbb{N}_0$,

$$\mathbf{M}^n = (M_{1,n}, \ldots, M_{p,n}),$$

where for all $i = 1, \ldots, p$, $(x_1, \ldots, x_p) \in I^p$,

$$M_{i,0}(x_1, \ldots, x_p) = x_i,$$

and for all $n \in \mathbb{N}_0$, $i = 1, \ldots, p$, $(x_1, \ldots, x_p) \in I^p$,

$$M_{i,n+1}(x_1, \ldots, x_p) = M_i\big(M_{1,n}(x_1, \ldots, x_p), \ldots, M_{p,n}(x_1, \ldots, x_p)\big).$$

Given a mean-type mapping $\mathbf{M}\colon I^p \to I^p$ and a mean $K\colon I^p \to I$, we say that [6] K is *invariant with respect to the mean-type mapping* $\mathbf{M}$, briefly $\mathbf{M}$-*invariant*, if

$$K \circ \mathbf{M} = K.$$

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Remark 1.6.
Note that a mean $K\colon I^p \to I$ is $\mathbf{M}$-*invariant* iff the mean-type mapping $\mathbf{K}\colon I^p \to I^p$ defined by $\mathbf{K} = (K, \dots, K)$ is $\mathbf{M}$-*invariant, that is, iff* $\mathbf{K} = \mathbf{K} \circ \mathbf{M}$.

Let us quote the following

Theorem 1.7 ([8]).
Let $I \subset \mathbb{R}$ be an interval and fix $p \in \mathbb{N}$, $p \geq 2$. Suppose that $\mathbf{M}\colon I^p \to I^p$, $\mathbf{M} = (M_1, \dots, M_p)$, is a continuous mean-type mapping of I^p such that, for all $(x_1, \dots, x_p) \in I^p \setminus \Delta(I^p)$,

$$\max\big(M_1(x_1, \dots, x_p), \dots, M_p(x_1, \dots, x_p)\big) - \min\big(M_1(x_1, \dots, x_p), \dots, M_p(x_1, \dots, x_p)\big) < \max(x_1, \dots, x_p) - \min(x_1, \dots, x_p).$$

Then

(i) *for every $n \in \mathbb{N}$, the n-th iterate $\mathbf{M}^n = (M_{1,n}, \dots, M_{p,n})$, is a mean-type mapping of I^p;*

(ii) *there is a continuous mean $K\colon I^p \to I$ such that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges uniformly on compact subsets of I^p to the mean-type mapping $\mathbf{K}\colon I^p \to I^p$, $\mathbf{K} = (K, \dots, K)$;*

(iii) *$\mathbf{K}\colon I^p \to I^p$ is $\mathbf{M}$-invariant, that is, $\mathbf{K} = \mathbf{K} \circ \mathbf{M}$, or, equivalently, the mean K is $\mathbf{M}$-invariant;*

(iv) *a continuous $\mathbf{M}$-invariant mean (mean-type mapping) is unique;*

(v) *if $\mathbf{M}$ is strict then so is K (and $\mathbf{K}$);*

(vi) *if $M_1, \dots, M_p$ are (strictly) increasing with respect to each variable then so is K;*

(vii) *if $I = (0, \infty)$ and $\mathbf{M}$ is positively homogeneous, then every iterate of $\mathbf{M}$ and K is positively homogeneous.*

Theorem 1.7 improves the relevant result in [5], where it is assumed that at most one of the coordinate means $M_1, \dots, M_p$ is not strict. In the case when $p = 2$ and the coordinate means are strict the suitable result is well known and frequently applied, cf. [2], [3, Chapter VI]. Let us also mention that the first paper concerning this subject is due to C.F. Gauss [4].

Remark 1.8.
The inequality assumed in Theorem 1.7 is indispensable.

To show it consider three examples.

Example 1.9.
Let $\mathbf{M}\colon I^p \to I^p$, $\mathbf{M} = (M_1, \dots, M_p)$, be a mean-type mapping such that

$$M_1(x_1, \dots, x_p) = \min(x_1, \dots, x_p), \qquad M_p(x_1, \dots, x_p) = \max(x_1, \dots, x_p).$$

Then, for all $x_1, \dots, x_p \in I$,

$$\begin{aligned}\min(x_1, \dots, x_p) + \max\big(M_1(x_1, \dots, x_p), \dots, M_p(x_1, \dots, x_p)\big) &= M_1(x_1, \dots, x_p) + M_p(x_1, \dots, x_p)\\ &= \min\big(M_1(x_1, \dots, x_p), \dots, M_p(x_1, \dots, x_p)\big) + \max(x_1, \dots, x_p),\end{aligned}$$

so the assumption of Theorem 1.7 is not fulfilled. Since the means $K = M_1$ and $K = M_p$ are $\mathbf{M}$-invariant, the uniqueness of the continuous $\mathbf{M}$-invariant mean fails.

Example 1.10.
Take $p = 3$, $I = \mathbb{R}$, and $\mathbf{M} = (M_1, M_2, M_3)$, where $M_1(x_1, x_2, x_3) = \min(x_1, x_2, x_3)$, $M_2(x_1, x_2, x_3) = x_1 + x_2 + x_3 - \min(x_1, x_2, x_3) - \max(x_1, x_2, x_3)$, $M_3(x_1, x_2, x_3) = \max(x_1, x_2, x_3)$. Then each of the means M_1, M_3 and $A_{[3]}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)/3$ is $\mathbf{M}$-invariant.

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Example 1.11.
Take $p = 2$, $I = \mathbb{R}$, and $\mathbf{M} = (M_1, M_2)$, where $M_1(x_1, x_2) = \min(x_1, x_2)$, and $M_2(x_1, x_2) = \max(x_1, x_2)$. Then each of the means M_1, M_2 and $A_{[2]}(x_1, x_2) = (x_1 + x_2)/2$ is $\mathbf{M}$-invariant.

In each of these examples the mean-type mapping $\mathbf{M}$ does not fulfil the basic assumption of Theorem 1.7 and the $\mathbf{M}$-invariant mean is not unique.

2. Explicit forms of invariant means for some classes of mean-type mappings

Now we consider some mean-type mappings for which the geometric mean is invariant. In the sequel we apply the convention that $\prod_{j=0}^{-1} a_j = 1$.

Proposition 2.1.
Let $p \in \mathbb{N}$, $p \geq 2$, be fixed. Then

(i) *for each $i = 1, \ldots, p$, the function $M_i \colon (0,\infty)^p \to (0,\infty)$ defined by*

$$M_i(x_1, \ldots, x_p) = \sum_{k=1}^{p} \prod_{j=0}^{i-1} x_{k+i+j-1} \Big/ \sum_{k=1}^{p} \prod_{j=0}^{i-2} x_{k+i+j-1}, \qquad \text{where} \quad x_{p+j} = x_j,$$

is a continuous and strict mean; in particular $M_1 = A_{[p]}$, $M_p = H_{[p]}$, where

$$A_{[p]}(x_1, \ldots, x_p) = \frac{1}{p} \sum_{k=1}^{p} x_k, \qquad \frac{1}{H_{[p]}(x_1, \ldots, x_p)} = \frac{1}{p} \sum_{k=1}^{p} \frac{1}{x_k};$$

(ii) *the geometric mean*

$$G_{[p]}(x_1, \ldots, x_p) = \sqrt[p]{x_1 \cdots x_p}$$

is $(M_1, \ldots, M_p)$-invariant, that is

$$G_{[p]} \circ (M_1, \ldots, M_p) = G_{[p]};$$

(iii) *the sequence of iterates $(M_1, \ldots, M_p)^n$, $n \in \mathbb{N}_0$, converges pointwise to the mean-type mapping $(G_{[p]}, \ldots, G_{[p]})$ in $(0,\infty)^p$; for all $x_1, \ldots, x_p > 0$,*

$$\lim_{n\to\infty} (M_1, \ldots, M_p)^n(x_1, \ldots, x_p) = \big(G_{[p]}(x_1, \ldots, x_p), \ldots, G_{[p]}(x_1, \ldots, x_p)\big).$$

Proof. To prove (i) it is enough to note that for each $i = 1, \ldots, p$, the function M_i is reflexive and strictly increasing with respect to each variable (we omit calculations). Part (ii) is obvious. Part (iii) follows from (i), (ii) and Theorem 1.7. □

Taking $p = 2$ in Proposition 2.1 we obtain

Corollary 2.2.
The geometric mean

$$G_{[2]}(x, y) = \sqrt{xy}, \qquad x, y > 0,$$

is $(A_{[2]}, H_{[2]})$-invariant, that is

$$G_{[2]} \circ (A_{[2]}, H_{[2]}) = G_{[2]},$$

and the sequence of iterates $(A_{[2]}, H_{[2]})^n$, $n \in \mathbb{N}_0$, converges pointwise to the mean-type mapping $(G_{[2]}, G_{[2]})$ in $(0,\infty)^2$, see [5, Proposition 1].

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Remark 2.3.

For $p = 2$, the invariance identity in (ii) generalizes the classical Pythagorean harmony proportion. Indeed, the relation $G_{[2]} \circ (A_{[2]}, H_{[2]}) = G_{[2]}$ is equivalent to

$$\frac{A_{[2]}}{G_{[2]}} = \frac{G_{[2]}}{H_{[2]}}.$$

Taking $p = 3$ in Proposition 2.1 we obtain

Corollary 2.4.

The geometric mean

$$G_{[3]}(x, y, z) = \sqrt[3]{xyz}, \qquad x, y, z > 0,$$

is $(A_{[3]}, M, H_{[3]})$-invariant, that is

$$G_{[3]} \circ (A_{[3]}, M, H_{[3]}) = G_{[3]},$$

and the sequence of iterates $(A_{[3]}, M, H_{[3]})^n$, $n \in \mathbb{N}_0$, converges pointwise to the mean-type mapping $(G_{[3]}, G_{[3]}, G_{[3]})$ in $(0, \infty)^3$. Here

$$M(x, y, z) = \frac{yz + zx + xy}{x + y + z}, \qquad x, y, z > 0.$$

Now we introduce a generalization of the weighted Bajraktarević means [9]. For $p \in \mathbb{N}$ let $\sigma_p \colon \{1, \dots, p\} \to \{1, \dots, p\}$ denote the cyclic permutation of the set $\{1, \dots, p\}$ defined by

$$\sigma_p(j) = j + 1, \qquad j \in \{1, \dots, p-1\}, \qquad \sigma_p(p) = 1,$$

and, for $i = 0, \dots, p-1$, by σ_p^i denote the i-th iterate of σ_p. Clearly,

$$\sigma_p^i(j) = i + j \quad \text{for} \quad j \in \{1, \dots, p-i\}, \qquad \sigma_p^i(j) = i + j - p \quad \text{for} \quad j \in \{p-i+1, \dots, p\}.$$

Thus $\sigma_p^1 = \sigma_p$ and σ_p^0 is the identity on the set $\{1, \dots, p\}$. From [9] we have the following

Proposition 2.5.

Let $p \in \mathbb{N}$, $p \geq 2$, $i \in \{0, 1, \dots, p-1\}$, and an interval $I \subset \mathbb{R}$ be fixed. Suppose that $f \colon I \to \mathbb{R}$ is continuous, strictly monotonic, and $g_1, \dots, g_p \colon I \to (0, \infty)$ are continuous. Then

(i) *for each $i = 1, \dots, p$, the function $B^{[f;\sigma_p^i]}_{[g_1,\dots,g_p]} \colon I^p \to \mathbb{R}$ defined by*

$$B^{[f;\sigma_p^i]}_{[g_1,\dots,g_p]}(x_1, \dots, x_p) = f^{-1}\left(\sum_{j=1}^{p} f(x_{\sigma_p^i}) g_j(x_j) \Big/ \sum_{j=1}^{p} g_j(x_j) \right)$$

is a strict continuous mean in I;

(ii) *the quasi-arithmetic mean $A_p^{[f]} \colon I^p \to I$ defined by*

$$A_p^{[f]}(x_1, \dots, x_p) = f^{-1}\left(\frac{1}{p} \sum_{j=1}^{p} f(x_j) \right)$$

is invariant with respect to the mean-type mapping

$$\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]} = \left(B^{[f;\sigma_p^0]}_{[g_1,\dots,g_p]}, B^{[f;\sigma_p^1]}_{[g_1,\dots,g_p]}, \dots, B^{[f;\sigma_p^{p-1}]}_{[g_1,\dots,g_p]} \right),$$

that is

$$A_p^{[f]} \circ \mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]} = A_p^{[f]};$$

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(iii) *the sequence* $\left(\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}\right)^n$, $n\in\mathbb{N}_0$, *of iterates of* $\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}$ *converges pointwise on* I^p *to the mean type mapping* $\mathbf{K}=\left(A^{[f]}_p,\dots,A^{[f]}_p\right)$.

Taking $i=0$ and $g_1=\ldots=g_p=g$ in the formula given in (i), we obtain $B^{[f;\sigma^0_p]}_{[g_1,\dots,g_p]}=B^{[f]}_{[g]}$, where

$$B^{[f]}_{[g]}(x_1,\dots,x_p)=f^{-1}\left(\sum_{j=1}^{p}f(x_j)g_j(x_j)\Big/\sum_{j=1}^{p}g_j(x_j)\right)$$

is the Bajraktarević mean [1].

Remark 2.6.

According to the second part of Proposition 2.5, the limit $\mathbf{K}$ of the sequence of iterates of the mean-type mapping $\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}$ (that is, the invariant mean-type mapping with respect to $\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}$) does not depend on the choice of the functions $g_1,\dots,g_p$.

Thus, taking for instance, $I=(0,\infty)$ and either $f(x)=x^p$ for some $p\in\mathbb{R}$, $p\neq 0$, or $f=\log$ and non-power functions $g_1,\dots,g_p$, one can obtain many non-homogeneous mean-type mappings $\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}$ with homogeneous invariant mean K. Applying Proposition 2.5 with $p=2$ we obtain the following

Corollary 2.7.

Let $I\subseteq\mathbb{R}$ *be an arbitrary interval. Suppose that* $f\colon I\to\mathbb{R}$ *is continuous, strictly monotonic, and* $g,h\colon I\to(0,\infty)$ *are continuous. Then*

(i) *the functions* $B^{[f]}_{[g,h]}, B^{[f]}_{[h,g]}\colon I^p\to\mathbb{R}$ *defined by*

$$B^{[f]}_{[g,h]}(x,y)=f^{-1}\left(\frac{f(x)g(x)+f(y)h(y)}{g(x)+h(y)}\right),\qquad B^{[f]}_{[g,h]}(x,y)=f^{-1}\left(\frac{f(x)h(y)+f(y)h(x)}{g(x)+h(y)}\right)$$

are strict continuous means on I;

(ii) *the quasi-arithmetic mean* $A^{[f]}_2\colon I^p\to I$ *defined by*

$$A^{[f]}_2(x,y)=f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$$

is invariant with respect to the mean-type mapping

$$\mathbf{B}^{[f]}_{g,h}=\left(B^{[f]}_{[g,h]},B^{[f]}_{[h,g]}\right),$$

that is

$$A^{[f]}_2\circ\mathbf{B}^{[f]}_{g,h}=A^{[f]}_2;$$

(iii) *the sequence* $\left(\mathbf{B}^{[f]}_{g,h}\right)^n$, $n\in\mathbb{N}_0$, *of iterates of* $\mathbf{B}^{[f]}_{g,h}$ *converges pointwise on* I^2 *to the mean type mapping* $\mathbf{K}=\left(A^{[f]}_2,A^{[f]}_2\right)$.

In the case $p=2$ we have also the following

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Proposition 2.8.
Let $I \subseteq \mathbb{R}$ be an arbitrary interval, $\varphi\colon I \to \mathbb{R}$ a continuous and strictly monotonic function, and $M\colon I^2 \to I$ an arbitrary continuous and strict mean. Then

(i) *the function $N\colon I^2 \to I$,*

$$N(x,y) = \varphi^{-1}\big(\varphi(x) + \varphi(y) - \varphi(M(x,y))\big),$$

is a continuous and strict mean in I;

(ii) *the quasi-arithmetic mean $A_2^{[\varphi]}$ is (M,N)-invariant;*

(iii) *the sequence of iterates $(M,N)^n$, $n \in \mathbb{N}_0$, converges pointwise to $\big(A_2^{[\varphi]}, A_2^{[\varphi]}\big)$ in I^2.*

Proof. To prove (i) assume that φ is strictly increasing. Take arbitrary $x, y \in I$. Without any loss of generality we can assume that $x \le y$. Then $\varphi(x) \le \varphi(M(x,y)) \le \varphi(y)$, whence

$$\varphi(x) \le \varphi(x) + \varphi(y) - \varphi(M(x,y)) \le \varphi(y).$$

By the Darboux property of the continuous function φ, it follows that $\varphi(x) + \varphi(y) - \varphi(M(x,y))$ belongs to the range of φ and, consequently, N is correctly defined. Since φ^{-1} is increasing, we have

$$\min(x,y) = x \le \varphi^{-1}\big(\varphi(x) + \varphi(y) - \varphi(M(x,y))\big) \le y = \max(x,y),$$

so the function N is a mean in I. In the case when φ is decreasing we argue similarly.

To prove (ii) note that, for all $x, y \in I$,

$$\big(A_2^{[\varphi]} \circ (M,N)\big)(x,y) = \varphi^{-1}\left(\frac{\varphi(M(x,y)) + \varphi[\varphi^{-1}(\varphi(x)+\varphi(y)-\varphi(M(x,y)))]}{2}\right) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) = A_2^{[\varphi]}(x,y).$$

Now part (iii) follows from Theorem 1.7. □

Taking $\varphi = \mathrm{id}|_I$ in Proposition 2.8 we obtain

Corollary 2.9.
Let $I \subseteq \mathbb{R}$ be an arbitrary interval and $M\colon I^2 \to I$ be an arbitrary continuous and strict mean. Then

(i) *the function $N\colon I^2 \to I$,*

$$N(x,y) = x + y - M(x,y),$$

is a continuous and strict mean in I;

(ii) *the arithmetic mean $A = A_{[2]}$ is (M,N)-invariant;*

(iii) *the sequence of iterates $(M,N)^n$, $n \in \mathbb{N}_0$, converges pointwise to (A,A) in I^2.*

Taking $\varphi = \log$ in Proposition 2.8 we obtain

Corollary 2.10.
Let $I \subseteq (0,\infty)$ be an interval and $M\colon I^2 \to I$ be a continuous and strict mean. Then

(i) *the function $N\colon I^2 \to I$,*

$$N(x,y) = \frac{xy}{M(x,y)},$$

is a continuous and strict mean in I;

(ii) *the geometric mean $G = G_{[2]}$ is (M,N)-invariant;*

(iii) *the sequence of iterates $(M,N)^n$, $n \in \mathbb{N}_0$, converges pointwise to (G,G) in I^2.*

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Taking $\varphi(x) = 1/x$ in Proposition 2.8 we obtain

Corollary 2.11.

Let $I \subseteq (0,\infty)$ be an interval and $M\colon I^2 \to I$ be a continuous and strict mean. Then

(i) *the function $N\colon I^2 \to I$,*

$$N(x,y) = \frac{xyM(x,y)}{(x+y)M(x,y) - xy},$$

is a continuous and strict mean in I;

(ii) *the harmonic mean $H = H_{[2]}$ is (M,N)-invariant;*

(iii) *the sequence of iterates $(M,N)^n$, $n \in \mathbb{N}_0$, converges pointwise to (H,H) in I^2.*

By the Lagrange mean-value theorem, if a real function f defined on an interval $I \subset \mathbb{R}$ is differentiable, then there exists a mean $L^{[f]}\colon I^2 \to I$ such that, for all $x, y \in I$, $x \neq y$,

$$\frac{f(x) - f(y)}{x - y} = f'\big(L^{[f]}(x,y)\big).$$

If moreover f' is one-to-one, then

$$L^{[f]}(x,y) = \begin{cases} (f')^{-1}\left(\dfrac{f(x) - f(y)}{x - y}\right) & \text{for} \quad x \neq y, \\ x & \text{for} \quad x = y, \end{cases}$$

so $L^{[f]}$ is strict and uniquely determined.

Remark 2.12.

For $a, b, c, d, p \in \mathbb{R}$, $acp \neq 0$, put

$$f(x) = ae^{px} + b, \qquad g(x) = ce^{-px} + d, \qquad x \in \mathbb{R}.$$

Then, cf. [7], $A \circ (L^{[f]}, L^{[g]}) = A$, where A is the arithmetic mean. Consequently, the sequence of iterates $(L^{[f]}, L^{[g]})^n$, $n \in \mathbb{N}_0$, converges pointwise to (A, A) in $\mathbb{R}^2$.

3. Some applications

Applying Proposition 2.5 we can prove the following, cf. [9],

Theorem 3.1.

Let $p \in \mathbb{N}$, $p \geq 2$, $i \in \{0, 1, \ldots, p-1\}$, and an interval $I \subset \mathbb{R}$ be fixed. Suppose that $f\colon I \to \mathbb{R}$ is continuous and strictly monotonic, and $g_1, \ldots, g_p\colon I \to (0,\infty)$ are continuous. Then a function $\Phi\colon I^p \to \mathbb{R}$, continuous on the diagonal $\Delta(I^p)$, satisfies the functional equation

$$\Phi\left(B^{[f;a_p^0]}_{[g_1,\ldots,g_p]}(x_1,\ldots,x_p), \ldots, B^{[f;a_p^{p-1}]}_{[g_1,\ldots,g_p]}(x_1,\ldots,x_p)\right) = \Phi(x_1,\ldots,x_p), \qquad x_1,\ldots,x_p \in I, \tag{1}$$

if and only if there exists a continuous function of a single variable $\varphi\colon I \to \mathbb{R}$ such that

$$\Phi(x_1,\ldots,x_p) = \varphi\left(f^{-1}\left(\frac{1}{p}\sum_{j=1}^{p} f(x_j)\right)\right), \qquad x_1,\ldots,x_p \in I,$$

(that is, if and only if there exists a continuous function $\varphi\colon I \to \mathbb{R}$ such that $\Phi = \varphi \circ A_p^{[f]}$).

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Indeed, if $\Phi\colon I^p \to \mathbb{R}$ satisfies equation (1), then

$$\Phi \circ \mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]} = \Phi,$$

and, by induction,

$$\Phi = \Phi \circ \left(\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}\right)^n, \qquad n \in \mathbb{N},$$

where $\left(\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}\right)^n$ denotes the n-th iterate of the mean-type mapping $\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}$. In view of Proposition 2.5, for all $x_1,\dots,x_p \in I$,

$$\lim_{n\to\infty}\left(\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}\right)^n(x_1,\dots,x_p) = \left(A^{[f]}_p(x_1,\dots,x_p),\dots,A^{[f]}_p(x_1,\dots,x_p)\right).$$

Hence, taking into account the continuity of Φ on the diagonal $\Delta(I^p)$, and setting

$$\varphi(t) = \Phi(t,\dots,t) \qquad \text{for} \quad t \in I,$$

we get

$$\begin{aligned}\Phi(x_1,\dots,x_p) = \lim_{n\to\infty}\Phi\circ\left(\mathbf{B}^{[f;\sigma_p]}_{[g_1,\dots,g_p]}\right)^n(x_1,\dots,x_p) &= \Phi\left(A^{[f]}_p(x_1,\dots,x_p),\dots,A^{[f]}_p(x_1,\dots,x_p)\right)\\ &= \varphi\left(A^{[f]}_p(x_1,\dots,x_p)\right) = \varphi\circ\left(A^{[f]}_p\right)(x_1,\dots,x_p)\end{aligned}$$

for all $x_1,\dots,x_p \in I$. (The "if" part is obvious.)

We omit writing analogous applications of Propositions 2.1 and 2.8, as well as of the corollaries of the previous section.

Remark 3.2.

From Corollary 2.2 we immediately get

$$\lim_{n\to\infty}(A_{[2]},H_{[2]})^n(2,1) = \sqrt{2}.$$

Since $(A_{[2]},H_{[2]})(x,y) = \left((x+y)/2,\, 2xy/(x+y)\right)$ and

$$(A_{[2]},H_{[2]})^2(x,y) = \left(\frac{x^2+6xy+y}{4(x+y)},\,\frac{4xy(x+y)}{x^2+6xy+y}\right),$$

we have

$$(A_{[2]},H_{[2]})^2(2,1) = \left(\frac{17}{12},\frac{24}{17}\right), \qquad \frac{24}{17} < \sqrt{2} < \frac{17}{12}, \qquad \text{and}$$

$$\sqrt{2}-\frac{24}{17} = 0.00244871\dots, \qquad \frac{17}{12}-\sqrt{2} = 0.00245324\dots$$

Thus the convergence of the sequence $(A_{[2]},H_{[2]})^n(2,1)$ is very fast.

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4. On nonexpansivity of mean-type mappings

In view of Corollary 2.2 the sequence of iterates $(A_{[2]}, H_{[2]})^n$, $n \in \mathbb{N}_0$, of the mean-type mapping $(A_{[2]}, H_{[2]})$ converges pointwise to the mean-type mapping $(G_{[2]}, G_{[2]})$ in $(0,\infty)^2$.

Remark 4.1.
The mean-type mapping $(A_{[2]}, H_{[2]})$ is not nonexpansive (with respect to the Euclidean norm).

To show this, for a fixed $a > 0$, put

$$\Gamma_a = \left\{ \left(s, \frac{a}{s} \right) : s > 0 \right\}.$$

Since, for all $s > 0$,

$$(A_{[2]}, H_{[2]}) \left(s, \frac{a}{s} \right) = \left(\frac{s + a/s}{2}, \frac{2sa/s}{s + a/s} \right) = \left(\frac{s^2 + a}{2s}, \frac{a}{(s^2 + a)/(2s)} \right),$$

$(A_{[2]}, H_{[2]})$ maps the hyperbola Γ_a into itself (thus the hyperbolas are invariant curves for the mapping $(A_{[2]}, H_{[2]})$, cf. [8]. Let us fix arbitrarily a point $\mathbf{p}_a = (s, a/s) \in \Gamma_a$, $s \neq \sqrt{a}$, and take a fixed $b > 0$, $b \neq a$. There is a unique point $\mathbf{p}_b = (t, b/t) \in \Gamma_b$, $t = t(s) > 0$, such that

$$\|\mathbf{p}_b - \mathbf{p}_a\| = \inf \{ \|\mathbf{p} - \mathbf{p}_a\| : \mathbf{p} \in \Gamma_b \}.$$

Now obviously we have

$$\left\| (A_{[2]}, H_{[2]})(\mathbf{p}_b) - (A_{[2]}, H_{[2]})(\mathbf{p}_a) \right\| \geq \|\mathbf{p}_b - \mathbf{p}_a\|.$$

For $a = 7/3$, $s = 1$ and $b = 6$ we easily get $t = t(1) = 2$, whence $\mathbf{p}_a = (1, 7/3)$, $\mathbf{p}_b = (2, 6)$, and

$$\left\| (A_{[2]}, H_{[2]})(\mathbf{p}_b) - (A_{[2]}, H_{[2]})(\mathbf{p}_a) \right\| = \frac{\sqrt{14}}{3} > \frac{\sqrt{13}}{3} = \|\mathbf{p}_b - \mathbf{p}_a\|,$$

which proves the remark.

If $\varphi \colon I \to \mathbb{R}$ is a continuous strictly monotonic function and $t \in (0,1)$, then the function $M_t^{[\varphi]} \colon I^2 \to \mathbb{R}$,

$$M_t^{[\varphi]}(x, y) = \varphi^{-1} \big(t\varphi(x) + (1-t)\varphi(y) \big),$$

where φ^{-1} denotes the inverse function of φ, is a strict mean in I, and it is called a quasi-arithmetic weighted mean, and t is called the weight of the mean, cf. for instance [3].

Now we prove the following

Theorem 4.2.
Let $\varphi \colon I \to \mathbb{R}$ be continuous and strictly monotonic, and let $t \in (0,1)$. The mean-type mapping $\left(M_t^{[\varphi]}, M_{1-t}^{[\varphi]} \right)$ is nonexpansive with respect to the Euclidean norm, i.e.

$$\left\| \left(M_t^{[\varphi]}, M_{1-t}^{[\varphi]} \right)(x, y) - \left(M_t^{[\varphi]}, M_{1-t}^{[\varphi]} \right)(u, v) \right\| \leq \|(x, y) - (u, v)\| \tag{2}$$

for all $(x, y), (u, v) \in I^2$, if and only if the function $f \colon \varphi(I) \times \varphi(I) \to \mathbb{R}$,

$$f(x, y) = \left(\varphi^{-1}(x) - \varphi^{-1}(y) \right)^2, \qquad x, y \in \varphi(I), \tag{3}$$

is convex.

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Proof. Condition (2) is equivalent to the inequality

$$\left[\varphi^{-1}\big(t\varphi(x)+(1-t)\varphi(y)\big)-\varphi^{-1}\big(t\varphi(u)+(1-t)\varphi(v)\big)\right]^2+\left[\varphi^{-1}\big((1-t)\varphi(x)+t\varphi(y)\big)-\varphi^{-1}\big((1-t)\varphi(u)+t\varphi(v)\big)\right]^2 \le (x-u)^2+(y-v)^2$$

for all $(x,y),(u,v)\in I^2$. Replacing here x,y,u,v respectively by $\varphi^{-1}(x)$, $\varphi^{-1}(y)$, $\varphi^{-1}(u)$, $\varphi^{-1}(v)$ for $x,y,u,v\in\varphi(I)$ we get

$$\left[\varphi^{-1}(tx+(1-t)y)-\varphi^{-1}(tu+(1-t)v)\right]^2+\left[\varphi^{-1}((1-t)x+ty)-\varphi^{-1}((1-t)u+tv)\right]^2 \le \left[\varphi^{-1}(x)-\varphi^{-1}(u)\right]^2+\left[\varphi^{-1}(y)-\varphi^{-1}(v)\right]^2$$

for all $(x,y),(u,v)\in[\varphi(I)]^2$. Using (3) we can write this inequality in the form

$$f\big(t(x,u)+(1-t)(y,v)\big)+f\big((1-t)(x,u)+t(y,v)\big)\le f(x,u)+f(y,v)$$

for all $(x,y),(u,v)\in[\varphi(I)]^2$, that is f is Wright-convex. In view of a theorem of Ng [10], the function f is the sum of a convex function and an additive function. The continuity of f implies that it is convex. □

In application of this result we need the following criterion of the convexity of function (3). (For convenience we write $h=\varphi^{-1}$.)

Proposition 4.3.
Let $J\subset\mathbb{R}$ be an interval and suppose that $h\colon J\to\mathbb{R}$ is twice differentiable in J and $h''\ge 0$ in J. Assume one the following two conditions is satisfied:

(i) *$h'>0$ in J and, for all $x,y\in J$, $x>y$,*

$$\frac{h''(x)}{[h'(x)]^2}-\frac{h''(y)}{[h'(y)]^2}\ge[h(x)-h(y)]\frac{h''(x)}{[h'(x)]^2}\frac{h''(y)}{[h'(y)]^2};$$

(ii) *$h'<0$ in J and, for all $x,y\in J$, $x>y$,*

$$\frac{h''(x)}{[h'(x)]^2}-\frac{h''(y)}{[h'(y)]^2}\le[h(x)-h(y)]\frac{h''(x)}{[h'(x)]^2}\frac{h''(y)}{[h'(y)]^2}.$$

Then the function $f\colon J^2\to\mathbb{R}$ given by

$$f(x,y)=[h(x)-h(y)]^2$$

is convex in J^2.

We omit an easy and straightforward proof. Now we apply Proposition 4.3.

Example 4.4.
Consider the function $h(x)=x^r$, $x>0$, for a fixed $r\in\mathbb{R}\setminus\{0\}$. Since

$$h'(x)=rx^{r-1},\qquad h''(x)=r(r-1)x^{r-2},$$

we have

$$\frac{h''(x)}{[h'(x)]^2}-\frac{h''(y)}{[h'(y)]^2}-[h(x)-h(y)]\frac{h''(x)}{[h'(x)]^2}\frac{h''(y)}{[h'(y)]^2}=\frac{1}{r^2x^ry^r}(y^r-x^r)(2r-1)(r-1).$$

Thus $h''\ge 0$ iff $r\ge 1$ or $r\le 0$. Now the first case in Proposition 4.3 occurs iff $r>0$ and $1/2\le r\le 1$, that is iff $r=1$. The second case cannot happen.

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Corollary 4.5.
Let $t \in (0,1)$ be fixed. The power mean mapping $T\colon (0,\infty)^2 \to (0,\infty)^2$,

$$T(x,y) = \left(tx^q + (1-t)y^q, (1-t)x^q + ty^q\right)$$

is nonexpansive if and only if $q = 1$. That is, iff T is the weighted arithmetic mean-type mapping

$$T(x,y) = (tx + (1-t)y, (1-t)x + ty).$$

Now we consider the nonexpansivity of the generalized weighted arithmetic mean-type mappings.

Theorem 4.6.
Let $\varphi\colon I \to \mathbb{R}$ be a strictly increasing function in an interval $I \subset \mathbb{R}$ such that

$$(0,\infty) \ni x \mapsto x - \varphi(x)$$

is strictly increasing. Then

(i) *φ is nonexpansive (so Lipschitzian) and the function $M\colon I^2 \to \mathbb{R}$ defined by*

$$M(x,y) = \varphi(x) + y - \varphi(y) \tag{4}$$

is a strict continuous mean in I;

(ii) *the mean-type mapping $T\colon I^2 \to I^2$,*

$$T(x,y) = (M(x,y), M(y,x))$$

is nonexpansive with respect to the Euclidean norm if and only if $\varphi(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \in (0,1)$, that is iff

$$M(x,y) = ax + (1-a)y. \tag{5}$$

Proof. The monotonicity of φ and (4) imply that, for all $x, y \in I$, $y \le x$, we have

$$0 \le \varphi(x) - \varphi(y) \le x - y,$$

which implies that φ is nonexpansive. The remaining part of the first result is obvious. To prove the second result assume that T is nonexpansive. Thus, for all $x, y, u, v \in I$,

$$\left[(\varphi(x) + y - \varphi(y)) - (\varphi(u) + v - \varphi(v))\right]^2 + \left[(\varphi(y) + x - \varphi(x)) - (\varphi(v) + u - \varphi(u))\right]^2 \le (x-u)^2 + (y-v)^2,$$

that is

$$\left[(\varphi(x) - \varphi(y) - \varphi(u) + \varphi(v)) + (y-v)\right]^2 + \left[(\varphi(y) - \varphi(x) - \varphi(v) + \varphi(u)) + (x-u)\right]^2 \le (x-u)^2 + (y-v)^2.$$

Putting

$$W = \varphi(x) - \varphi(y) - \varphi(u) + \varphi(v)$$

we have

$$[W + (y-v)]^2 + [-W + (x-u)]^2 \le (x-u)^2 + (y-v)^2,$$

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which (after simple calculations) reduces to the inequality

$$W[W-(x-u+v-y)]\leq 0.$$

Thus $0\leq W\leq x-u+v-y$, whence, by the the definition of W,

$$0\leq \varphi(x)-\varphi(y)-\varphi(u)+\varphi(v)\leq x-u+v-y$$

for all $x,y,u,v\in I$. Setting here $u=y=(x+v)/2$ we get

$$0\leq \varphi(x)-2\varphi\left(\frac{x+v}{2}\right)+\varphi(v)\leq 0$$

whence, for all $x,v\in I$,

$$\varphi\left(\frac{x+v}{2}\right)=\frac{\varphi(x)+\varphi(v)}{2}.$$

The continuity of φ implies that it is affine, i.e. there exist $a,b\in\mathbb{R}$ such that $\varphi(x)=ax+b$ for all $x\in I$, where $a\neq 0$. It follows that (5) holds true. The converse implication is obvious. □

The results of this section show that, in general, the existence of fixed points and the convergence of iterates does not follow from the fixed point theory of nonexpansive (or contractive) mappings.

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