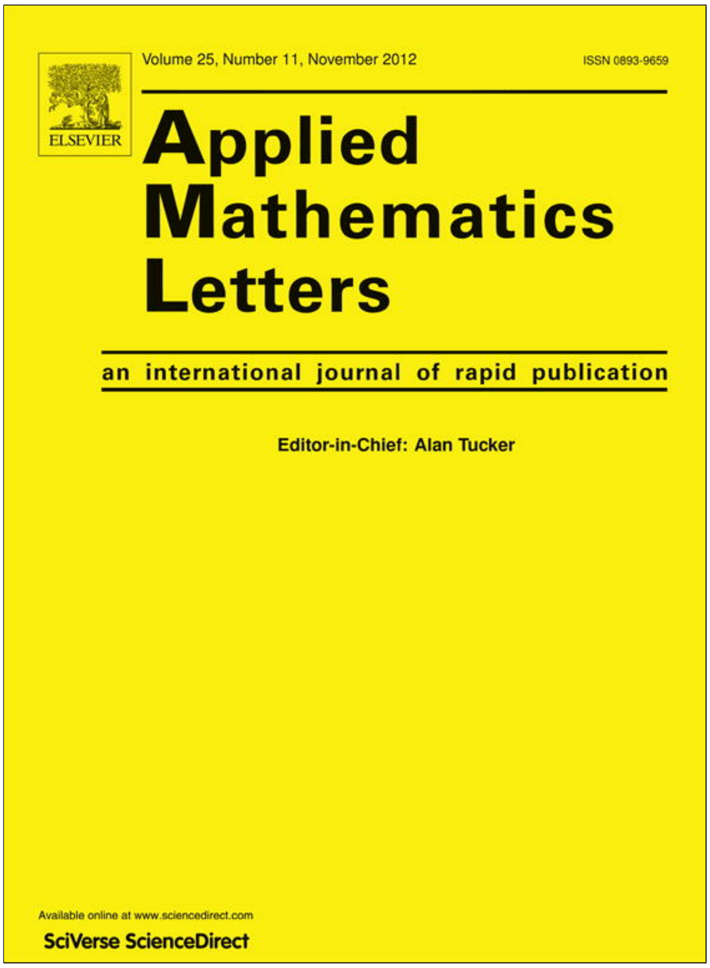
Volume 25, Number 11, November 2012
ISSN 0893-9659
ELSEVIER
Applied Mathematics Letters
an international journal of rapid publication
Editor-in-Chief: Alan Tucker
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Applied Mathematics Letters 25 (2012) 1651–1655

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Invariance of a quasi-arithmetic mean with respect to a system of generalized Bajraktarević means

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ARTICLE INFO

Article history:
Received 21 April 2011
Received in revised form 7 January 2012
Accepted 21 January 2012

Keywords:
Mean
Invariant mean
Quasi-arithmetic mean
Bajraktarević mean
Generalized Bajraktarević mean
Functional equation

ABSTRACT

Let a positive integer $k \geq 2$ and an interval $I \subset \mathbb{R}$ be fixed. For a continuous strictly monotonic function $f : I \to \mathbb{R}$, and arbitrary continuous function $g_1, \ldots, g_k : I \to (0, \infty)$, we define a system of means $B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^i]} : I^k \to I$ for $i \in \{0, 1, \ldots, k-1\}$, where σ_k^i is the ith iterate of a cycle permutation of the variables. These means generalize the Bajraktarević means. We show that the quasi-arithmetic mean of the generator f is invariant with respect to the mean-type mapping of this system. The effective formula for the limit of the iterates of these mean-type mappings is given. An application in solving a functional equation is presented.

1. Introduction

Let a positive integer $k \geq 2$ and an interval $I \subseteq \mathbb{R}$ be fixed. A function $M : I^k \to \mathbb{R}$ is called a *mean* in an interval I (cf. [1]) if

$$\min(x_1, \ldots, x_k) \leq M(x_1, \ldots, x_k) \leq \max(x_1, \ldots, x_k), \qquad x_1, \ldots, x_k \in I.$$

Denote by $\Delta(I^k)$ the diagonal of I^k, that is

$$\Delta(I^k) := \{(x_1, \ldots, x_k) \in I^k : x_1 = \cdots = x_k\}.$$

A mean M is called *strict* if, for all $(x_1, \ldots, x_k) \in I^k \setminus \Delta(I^k)$, the above inequalities are sharp, and *symmetric* if, for any permutation $(i_1, \ldots, i_k)$ of the set $\{1, \ldots, k\}$,

$$M(x_{i_1}, \ldots, x_{i_k}) = M(x_1, \ldots, x_k).$$

A mean $M : (0, \infty)^k \to (0, \infty)$ is said to be *positively homogeneous* if

$$M(tx_1, \ldots, tx_k) = tM(x_1, \ldots, x_k), \quad t, x_1, \ldots, x_k > 0.$$

If M is a mean in I then $M(J^k) = J$ for every subinterval $J \subseteq I$. Moreover M is *reflexive*, i.e.

$$M(x, \ldots, x) = x, \quad x \in I.$$

Every reflexive function $M : I^k \to \mathbb{R}$ that is increasing with respect to each variable is a mean in I.

Let $M_1, \ldots, M_k : I^k \to I$ be means. A mean $K : I^k \to I$ is called *invariant with respect to the mean-type mapping* $\mathbf{M} := (M_1, \ldots, M_k) : I^k \to I^k$ (for short, K is $\mathbf{M}$-*invariant*), if

$$K(M_1(x_1, \ldots, x_k), \ldots, M_k(x_1, \ldots, x_k)) = K(x_1, \ldots, x_k), \quad x_1, \ldots, x_k \in I,$$

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2012.01.030

or, for short, if $K \circ \mathbf{M} = K$ (cf. [2]). The mean K is also referred to as the Gauss composition of the means $M_1, \ldots, M_k$ (cf. [3]). The invariant mean is useful when we are looking for the limits of the sequence of iterates of the mean-type mapping $\mathbf{M} : I^k \to I^k$ (cf. [4–7]). Let us quote the following:

Theorem 1 *([7]). Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 2$, fixed. Suppose that $\mathbf{M} : I^k \to I^k$, $\mathbf{M} = (M_1, \ldots, M_k)$, is a continuous mean-type mapping such that, for all $(x_1, \ldots, x_k) \in I^k \setminus \Delta(I^k)$,*

$$\begin{aligned}&\max(M_1(x_1, \ldots, x_k), \ldots, M_k(x_1, \ldots, x_k)) - \min(M_1(x_1, \ldots, x_k), \ldots, M_k(x_1, \ldots, x_k))\\ &\quad < \max(x_1, \ldots, x_k) - \min(x_1, \ldots, x_k).\end{aligned}$$

Then:

(1) *for every $n \in \mathbb{N}$, the nth iterate $\mathbf{M}^n = (M_{n,1}, \ldots, M_{n,k})$ is a mean-type mapping of I^k;*
(2) *there is a continuous mean $K : I^k \to I$ such that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges, uniformly on compact subsets of I^k, to the mean-type mapping $\mathbf{K} : I^k \to I^k$, $\mathbf{K} = (K_1, \ldots, K_k)$, such that*

$$K_1 = \cdots = K_k = K;$$

(3) $\mathbf{K} : I^k \to I^k$ *is* $\mathbf{M}$-invariant, that is,

$$\boldsymbol{K} = \boldsymbol{K} \circ \boldsymbol{M}$$

or, equivalently, the mean K is $\mathbf{M}$-invariant*;*
(4) *the continuous M-invariant mean K is unique;*
(5) *if $\mathbf{M}$ is strict then so is K (and $\mathbf{K}$);*
(6) *if $M_1, \ldots, M_k$ are (strictly) increasing with respect to each variable then so is K;*
(7) *if $I = (0, \infty)$ and $\mathbf{M}$ is positively homogeneous, then all iterates of $\mathbf{M}$ and K are positively homogeneous.*

Using this result, we determine an effective form of the invariant means for a broad class of generalized Bajraktarević mean-type mappings. We apply this fact in solving a functional equation.

2. Generalized weighted Bajraktarević means

For $k \in \mathbb{N}$ let $\sigma_k : \{1, \ldots, k\} \to \{1, \ldots, k\}$ denote the cyclic permutation of the set $\{1, \ldots, k\}$ defined by

$$\sigma_k(j) := j + 1 \quad \text{for } j \in \{1, \ldots, k-1\} \quad \text{and} \quad \sigma_k(k) := 1,$$

and, for $i = 0, \ldots, k-1$, by σ_k^i denote the ith iterate of σ_k. Clearly,

$$\sigma_k^i(j) = i + j \quad \text{for } j \in \{1, \ldots, k-i\}; \qquad \sigma_k^i(j) = i + j - k, \quad \text{for } j \in \{k-i+1, \ldots, k\}.$$

Thus $\sigma_k^1 = \sigma_k$ and σ_k^0 is the identity of the set $\{1, \ldots, k\}$.

Theorem 2. *Let $k \in \mathbb{N}$, $k \geq 2$; $i \in \{0, 1, \ldots, k-1\}$, and an interval $I \subset \mathbb{R}$ be fixed. If $f : I \to \mathbb{R}$ is continuous strictly monotonic, and $g_1, \ldots, g_k : I \to (0, \infty)$ are continuous, then the function $B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^i]} : I^k \to \mathbb{R}$ defined by*

$$B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^i]}(x_1, \ldots, x_k) := f^{-1}\left(\frac{\sum_{j=1}^{k} f(x_{\sigma_k^i(j)})g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)}\right) \tag{2.1}$$

is a strict continuous mean in I.

Proof. Without any loss of generality we may assume that f is strictly increasing. Take arbitrary $x_1, \ldots, x_k \in I$ and put $x := \min(x_1, \ldots, x_k)$, $y := \max(x_1, \ldots, x_k)$. Then

$$f(x) = \frac{\sum_{j=1}^{k} f(x)g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)} \leq \frac{\sum_{j=1}^{k} f(x_{\sigma_k^i(j)})g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)} \leq \frac{\sum_{j=1}^{k} f(y)g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)} = f(y)$$

and, clearly, these inequalities are sharp if $x < y$. Hence, taking into account that f is strictly increasing, by (2.1) we get

$$\min(x_1, \ldots, x_k) = x \leq B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^i]}(x_1, \ldots, x_k) \leq y = \max(x_1, \ldots, x_k),$$

and these inequalities are sharp if $x < y$. Thus $M_{[g_1,\ldots,g_k]}^{[f;\sigma_k^i]}$ is a strict mean in I. Since the continuity of $B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^i]}$ is obvious, the proof is complete. □

Remark 1. Taking $i = 0$ and $g_1 = \cdots = g_k = g$ in formula (2.1), we obtain $B^{[f;\sigma_k^0]}_{[g_1,\ldots,g_k]} = B^{[f]}_{[g]}$, where

$$B^{[f]}_{[g]}(x_1, \ldots, x_k) := f^{-1}\left(\frac{\sum_{j=1}^{k} f(x_j)g(x_j)}{\sum_{j=1}^{k} g(x_j)}\right) \tag{2.2}$$

is the Bajraktarević mean (cf. [8]). Thus the mean (2.1) is a generalization of the Bajraktarević mean.

Remark 2. If $k \geq 3$, the means $B^{[f;\sigma_k^i]}_{[g_1,\ldots,g_k]}$ for $i \in \{1, \ldots, k-1\}$ need not be of the Bajraktarević type.

Indeed, taking for instance $k = 3$, $I = (0, \infty)$, $f(x) = x$, $g_1(x) = g_2(x) = g_3(x) = e^x$, one can show that

$$B^{[f;\sigma_3^1]}_{[g_1,g_2,g_3]}(x, y, z) = \frac{ze^x + xe^y + ye^z}{e^x + e^y + e^z}; \qquad B^{[f;\sigma_3^2]}_{[g_1,g_2,g_3]}(x, y, z) = \frac{ye^x + ze^y + xe^z}{e^x + e^y + e^z},$$

are not of the Bajraktarević type.

In the case when $k = 2$, the situation is different. Namely we have the following:

Remark 3. If $k = 2$ and $g_1 = g_2 = g$, then both $B^{[f;\sigma_2^0]}_{[g_1,g_2]}$ and $B^{[f;\sigma_2^1]}_{[g_1,g_2]}$ are members of the family of Bajraktarević means.

Indeed, in the case $B^{[f;\sigma_2^0]}_{[g_1,g_2]}$ this is obvious (cf. Remark 1), and for all $x, y \in I$ we have

$$\begin{aligned} B^{[f;\sigma_2^1]}_{[g_1,g_2]}(x, y) &= f^{-1}\left(\frac{f(y)g(x) + f(x)g(y)}{g(x) + g(y)}\right) = f^{-1}\left(\frac{f(x)\frac{1}{g(x)} + f(y)\frac{1}{g(y)}}{\frac{1}{g(x)} + \frac{1}{g(y)}}\right) \\ &= B^{[f]}_{[1/g]}(x, y). \end{aligned}$$

Theorem 3. *Suppose that the assumptions of Theorem 2 are satisfied. Then:*

(i) *the quasi-arithmetic mean $A^{[f]}_k : I^k \to I$ defined by*

$$A^{[f]}_k(x_1, \ldots, x_k) := f^{-1}\left(\frac{1}{k}\sum_{j=1}^{k} f(x_j)\right) \tag{2.3}$$

is invariant with respect to the mean-type mapping

$$\mathbf{B}^{[f;\sigma_k]}_{[g_1,\ldots,g_k]} := (B^{[f;\sigma_k^0]}_{[g_1,\ldots,g_k]}, B^{[f;\sigma_k^1]}_{[g_1,\ldots,g_k]}, \ldots, B^{[f;\sigma_k^{k-1}]}_{[g_1,\ldots,g_k]}),$$

that is

$$A^{[f]}_k \circ \mathbf{B}^{[f;\sigma_k]}_{[g_1,\ldots,g_k]} = A^{[f]}_k; \tag{2.4}$$

(ii) *the sequence $((\mathbf{B}^{[f;\sigma_k]}_{[g_1,\ldots,g_k]})^n : n \in \mathbb{N}_0)$ of iterates of $\mathbf{B}^{[f;\sigma_k]}_{[g_1,\ldots,g_k]}$ converges, uniformly on compact subsets of I^k, to the mean-type mapping $\mathbf{K} = (K_1, \ldots, K_k)$ such that*

$$K_j = A^{[f]}_k, \quad j = 1, \ldots, k.$$

Proof. To prove (i) take arbitrary $(x_1, \ldots, x_k) \in I^k$. From the definitions (2.1), (2.3), making simple calculations, we obtain

$$\begin{aligned} A^{[f]}_k \circ \mathbf{B}^{[f;\sigma_k]}_{[g_1,\ldots,g_k]}(x_1, \ldots, x_k) &= f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} f(B^{[f;\sigma_k^{i-1}]}_{[g_1,\ldots,g_k]}(x_1, \ldots, x_k))\right) \\ &= f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} f\left(f^{-1}\left(\frac{\sum_{j=1}^{k} f(x_{\sigma_k^{i-1}(j)})g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)}\right)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{\sum_{j=1}^{k} f(x_{\sigma_k^{i-1}(j)})g_j(x_j)}{\sum_{j=1}^{k} g_j(x_j)}\right)\right)\\
&= f^{-1}\left(\frac{1}{k\sum_{j=1}^{k} g_j(x_j)}\sum_{i=1}^{k}\left(\sum_{j=1}^{k} f(x_{\sigma_k^{i-1}(j)})g_j(x_j)\right)\right)\\
&= f^{-1}\left(\frac{1}{k\sum_{j=1}^{k} g_j(x_j)}\sum_{j=1}^{k}\left(\sum_{i=1}^{k} f(x_{\sigma_k^{i-1}(j)})g_j(x_j)\right)\right)\\
&= f^{-1}\left(\frac{1}{k\sum_{j=1}^{k} g_j(x_j)}\sum_{j=1}^{k}\left(\sum_{i=1}^{k} f(x_{\sigma_k^{i-1}(j)})\right)g_j(x_j)\right)\\
&= f^{-1}\left(\frac{1}{k\sum_{j=1}^{k} g_j(x_j)}\left(\sum_{j=1}^{k} g_j(x_j)\right)\left(\sum_{i=1}^{k} f(x_{\sigma_k^{i-1}})\right)\right)\\
&= f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} f(x_{\sigma_k^{i-1}})\right) = f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} f(x_i)\right)\\
&= A_k^{[f]}(x_1,\ldots,x_k),
\end{aligned}
$$

which proves (2.4).

Part (ii) follows from Theorem 1. □

Remark 4. It is surprising that, according to the second part of Theorem 3, the limit $\mathbf{K}$ of the sequence of iterates of the mean-type mapping $\mathbf{B}_{[g_1,\ldots,g_k]}^{[f;\sigma_k]}$ (that is the invariant mean-type mapping with respect to $\mathbf{B}_{[g_1,\ldots,g_k]}^{[f;\sigma_k]}$) does not depend on the choice of the functions $g_1,\ldots,g_k$.

Thus, taking, for instance, $I=(0,\infty)$ and either $f(x)=x^p$ for some $p\in\mathbb{R}$, $p\neq 0$, or $f=\log$ and non-power functions $g_1,\ldots,g_k$, one can obtain a lot of non-homogeneous mean-type mappings $\mathbf{B}_{[g_1,\ldots,g_k]}^{[f;\sigma_k]}$ with homogeneous invariant mean K.

3. An application

Theorem 4. *Let $k\in\mathbb{N}$, $k\geq 2$; $i\in\{0,1,\ldots,k-1\}$, and an interval $I\subset\mathbb{R}$ be fixed. Suppose that $f: I\to\mathbb{R}$ is continuous, strictly monotonic, and $g_1,\ldots,g_k: I\to(0,\infty)$ are continuous. Then a function $\Phi: I^k\to\mathbb{R}$, continuous on the diagonal $\Delta(I^k)$, satisfies the functional equation*

$$\Phi\left(B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^0]}(x_1,\ldots,x_k),\ldots,B_{[g_1,\ldots,g_k]}^{[f;\sigma_k^{k-1}]}(x_1,\ldots,x_k)\right)=\Phi(x_1,\ldots,x_k),\quad x_1,\ldots,x_k\in I, \tag{3.1}$$

if, and only if, there exists a continuous function of a single variable $\varphi: I\to\mathbb{R}$ such that

$$\Phi(x_1,\ldots,x_k)=\varphi\left(f^{-1}\left(\frac{1}{k}\sum_{j=1}^{k} f(x_j)\right)\right),\quad x_1,\ldots,x_k\in I \tag{3.2}$$

(that is, if, and only if, there exists a continuous function of a single variable $\varphi: I\to\mathbb{R}$ such that

$$\Phi=\varphi\circ A_k^{[f]}.)$$

Proof. Assume that a function $\Phi: I^k\to\mathbb{R}$ is continuous on the diagonal $\Delta(I^k)$ and satisfies (3.1). Then we have

$$\Phi\circ\mathbf{B}_{[g_1,\ldots,g_k]}^{[f;\sigma_k]}=\Phi,$$

and, by induction,

$$\Phi = \Phi \circ (\mathbf{B}^{[f;\sigma_k]}_{[g_1,\dots,g_k]})^n, \quad n \in \mathbb{N},$$

where $(\mathbf{B}^{[f;\sigma_k]}_{[g_1,\dots,g_k]})^n$ denotes the nth iterate of the mean-type mapping $\mathbf{B}^{[f;\sigma_k]}_{[g_1,\dots,g_k]}$. In view of Theorem 3, for all $x_1, \dots, x_k \in I$,

$$\lim_{n\to\infty} (\mathbf{B}^{[f;\sigma_k]}_{[g_1,\dots,g_k]})^n (x_1, \dots, x_k) = (K_1(x_1, \dots, x_k), \dots, K_k(x_1, \dots, x_k)),$$

where

$$K_j = A^{[f]}_k, \quad j = 1, \dots, k.$$

Hence, taking into account the continuity of Φ on the diagonal $\Delta(I^k)$, and setting

$$\varphi(t) := \Phi(t, \dots, t), \quad \text{for } t \in I,$$

we obtain, for all $x_1, \dots, x_k \in I$,

$$\begin{aligned}\Phi(x_1, \dots, x_k) &= \lim_{n\to\infty} \Phi \circ (\mathbf{B}^{[f;\sigma_k]}_{[g_1,\dots,g_k]})^n (x_1, \dots, x_k) \\ &= \Phi(A^{[f]}_k(x_1, \dots, x_k), \dots, A^{[f]}_k(x_1, \dots, x_k)) \\ &= \varphi(A^{[f]}_k(x_1, \dots, x_k)) = \varphi \circ (A^{[f]}_k)(x_1, \dots, x_k),\end{aligned}$$

which completes the "only if" part of the theorem. Since the "if" part is obvious, the proof is complete. □

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