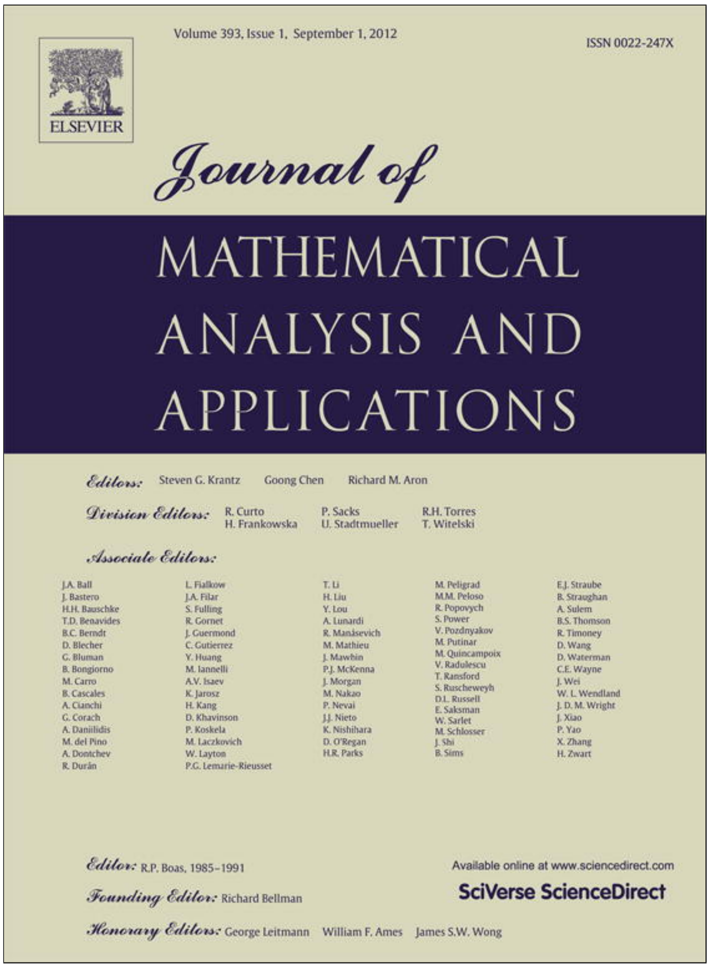
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The Pexider type generalization of the Minkowski inequality

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ABSTRACT

Let (Ω, Σ, μ) be a measure space such that $0 < \mu(A) < 1 < \mu(B) < \infty$ for some $A, B \in \Sigma$. The following converse Minkowski inequality theorem is proved in Matkowski (2008) [4]. If $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective, φ is increasing, and

$$\varphi^{-1}\left(\int_{\Omega(\mathbf{x}+\mathbf{y})} \varphi \circ (\mathbf{x}+\mathbf{y}) d\mu\right) \leq \psi^{-1}\left(\int_{\Omega(\mathbf{x})} \psi \circ \mathbf{x} d\mu\right) + \gamma^{-1}\left(\int_{\Omega(\mathbf{y})} \gamma \circ \mathbf{y} d\mu\right)$$

for all nonnegative μ-integrable simple functions $\mathbf{x}, \mathbf{y} : \Omega \to \mathbb{R}$ (where $\Omega(\mathbf{x})$ stands for the support of $\mathbf{x}$), then there exists a real $p \geq 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\psi(t)}{\psi(1)} = \frac{\gamma(t)}{\gamma(1)} = t^p.$$

In the present paper we show that if, in the basic measure space, there is no $A \in \Sigma$ such that either $1 < \mu(A) < \infty$ or $0 < \mu(A) < 1$, then there are some broad classes of non-power functions which satisfy the above Minkowski type inequality. Moreover we prove that, in the converse of the Minkowski inequality theorem, the assumption of the increasing monotonicity of φ is essential.

0. Introduction

For a measure space (Ω, Σ, μ), $S = S(\Omega, \Sigma, \mu)$ denotes the linear real space of all μ-integrable simple functions $\mathbf{x} : \Omega \to \mathbb{R}$. Let $S_+ := \{\mathbf{x} \in S : \mathbf{x} \geq 0\}$. For the support of $\mathbf{x} \in S$ put

$$\Omega(\mathbf{x}) := \{\omega \in \Omega : \mathbf{x}(\omega) \neq 0\}.$$

Let $\varphi : (0, \infty) \to (0, \infty)$ be an arbitrary bijection (that is one-to-one and onto: $\varphi((0, \infty)) = (0, \infty)$). Since, for any $\mathbf{x} \in S_+$, the restriction $\varphi \circ \mathbf{x}\big|_{\Omega(\mathbf{x})} \in S_+$, the functional $\mathbb{P}_\varphi : S \to [0, \infty)$ given by the formula

$$\mathbb{P}_\varphi(\mathbf{x}) := \begin{cases} \varphi^{-1}\left(\int_{\Omega(\mathbf{x})} \varphi \circ |\mathbf{x}|\, d\mu\right) & \text{if } \mu(\Omega(\mathbf{x})) > 0 \\ 0 & \text{if } \mu(\Omega(\mathbf{x})) = 0, \end{cases}$$

is correctly defined.

By the Minkowski inequality:

if $\varphi(t) = \varphi(1)t^p$ *for some* $p \geq 1$, *then*

$$\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y}) \leq \mathbb{P}_\varphi(\mathbf{x}) + \mathbb{P}_\varphi(\mathbf{y}), \quad x, y \in S_+(\Omega, \Sigma, \mu). \tag{M}$$

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In [1] (cf. also [2]) it has been shown that, in general, under some rather weak regularity assumptions of φ (or without any), the converse implication holds if,

$$\text{there are } A, B \in \Sigma \quad \text{such that } 0 < \mu(A) < 1 < \mu(B) < \infty. \tag{$*$}$$

(If the range of the measure is enough rich, no regularity assumption is needed [3].)

In [4] the following result has been proved. *Suppose that there are $A, B \in \Sigma$ such that $(*)$ holds true and $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective. If φ is increasing, then*

$$\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y}) \le \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu), \tag{P}$$

if, and only if, there exists a real $p \ge 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\psi(t)}{\psi(1)} = \frac{\gamma(t)}{\gamma(1)} = t^p, \quad t > 0.$$

Inequality (P) is called a "Pexiderization" of the Minkowski inequality (M) as, for the first time, an analogous procedure was applied in 1903 by Pexider [5] for the Cauchy functional equation. Let us note that, under the basic condition $(*)$, and the increasing monotonicity of φ, the above converse of the Minkowski inequality theorem remains true, if (P) is replaced by the inequality

$$\mathbb{P}_\varphi\left(\sum_{i=1}^n \mathbf{x}_i\right) \le \sum_{i=1}^n \mathbb{P}_{\psi_i}(\mathbf{x}_i), \quad \mathbf{x}_i \in S_+(\Omega, \Sigma, \mu),\ i = 1, \dots, n,$$

where $n \in, \mathbb{N}, n \ge 2$, is arbitrarily fixed (cf. [4]).

In Section 2 we show that, in these results, the assumption of the increasing monotonicity of φ is indispensable.

Condition $(*)$ plays here a crucial role. Note that the measure space (Ω, Σ, μ) does not satisfy $(*)$, iff either

(I) *for every $A \in \Sigma$, we have $\mu(A) = 0$ or $\mu(A) \ge 1$;*

or

(II) *for every $A \in \Sigma$, we have $\mu(A) \le 1$ or $\mu(A) = \infty$.*

In the present paper, in each of these two cases we indicate some broad classes of *non-power functions* satisfying inequality (P). In case (I), the results obtained in Section 3 extend an earlier Mulholland inequality [6] (cf. also [7,8]) as well as its generalizations [1,2,4] concerning inequality (M). It turns out that the convexity and geometrical convexity of φ plays a meaningful role.

Section 4 is devoted to the "Pexiderized" Minkowski inequality in case (II). In particular we show that if (Ω, Σ, μ) is a non-trivial probability measure space, inequality (P) is equivalent to concavity of the function

$$(0, \infty)^2 \ni (s, t) \longmapsto \varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right).$$

1. Preliminaries

Let (Ω, Σ, μ) be a measure space. Denote by χ_A the characteristic function of a set A. For $A, B \in \Sigma, A \cap B = \emptyset$, we put

$$S_+(A, B) := \{x_1\chi_A + x_2\chi_B \in S_+ : x_1, x_2 > 0\}$$

and

$$a := \mu(A), \qquad b := \mu(B).$$

Let $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ be arbitrary bijections. If $\mathbf{x} := x_1\chi_A + x_2\chi_{B\setminus A} \in S_+(A, B)$ then, by the definition of $\mathbb{P}_\varphi$, we have

$$\mathbb{P}_\varphi(x) = \varphi^{-1}\left(a\varphi(x_1+y_1) + b\varphi(x_2+y_2)\right).$$

It follows that the inequality

$$\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y}) \le \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(A, B), \tag{1.1}$$

where

$$\mathbf{x} := x_1\chi_A + x_2\chi_{B\setminus A}, \qquad \mathbf{y} := y_1\chi_A + y_2\chi_{B\setminus A},$$

for some $x_1, x_2, y_1, y_2 > 0$, (that is, inequality (P) restricted to the set $S_+(A, B)$) can be written in the following more explicit form:

$$\varphi^{-1}\left(a\varphi(x_1+y_1) + b\varphi(x_2+y_2)\right) \le \psi^{-1}\left(a\psi(x_1) + b\psi(x_2)\right) + \gamma^{-1}\left(a\gamma(y_1) + b\gamma(y_2)\right),$$
$$x_1, x_2, y_1, y_2 > 0. \tag{1.2}$$

We shall need the following obvious.

Remark 1. Let A, B be arbitrary disjoint nonempty sets and $a, b > 0$ real numbers. Then (Ω, Σ, μ) with $\Omega := A \cup B$, $\Sigma = \{\emptyset, A, B, A \cup B\}$ and μ defined by $\mu(A) := a$, $\mu(B) := b$ is a measure space. Moreover $S_+(\Omega, \Sigma, \mu) = S_+(A, B)$ and inequality (1.2) (as well as (1.1) is equivalent to the inequality)

$$\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y}) \le \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu).$$

2. Increasing monotonicity of φ in the converse of the Minkowski inequality theorem is essential

Let us quote the following [4] (Theorem 2).

Theorem 1. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that condition $(*)$ is fulfilled. Suppose that $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective functions and φ is strictly increasing. Then inequality (P) holds if, and only if, there is $p \geq 1$ such that*

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\psi(t)}{\psi(1)} = \frac{\gamma(t)}{\gamma(1)} = t^p, \quad t > 0.$$

To show that the assumption of the increasing monotonicity of φ in this converse of the Minkowski inequality is essential we prove the following.

Proposition 1. *Let the real numbers $a, b > 0$ be arbitrarily fixed. If the bijections $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ such that*

(i) *φ is decreasing and convex;*
(ii) *ψ and γ are increasing and convex;*
(iii) *the functions*

$$f := \varphi^{-1} \circ ((a+b)\varphi), \qquad g := \psi^{-1} \circ ((a+b)\psi), \qquad h := \gamma^{-1} \circ ((a+b)\gamma)$$

satisfy the Pexider type subadditivity condition

$$f(s+t) \leq g(s) + h(t), \quad s, t > 0,$$

then inequality (1.2) holds true.

Proof. Replacing s by $\psi^{-1}(s)$, t by $\gamma^{-1}(t)$, and making use of decreasing monotonicity of φ^{-1}, we can write condition (iii) in the following equivalent form

$$\varphi\left(\psi^{-1}\left((a+b)s\right) + \gamma^{-1}\left((a+b)t\right)\right) \leq (a+b)\varphi(\psi^{-1}(s) + \gamma^{-1}(t)), \quad s, t > 0.$$

By (ii) the functions ψ^{-1} and γ^{-1} are concave. It follows that the functions

$$(0, \infty) \ni u \longmapsto \psi^{-1}\left((a+b)u\right), \qquad (0, \infty) \ni u \longmapsto \gamma^{-1}\left((a+b)u\right)$$

are also concave. Therefore

$$\begin{aligned}
\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_1 + by_1) &= \psi^{-1}\left((a+b)\left(\frac{a}{a+b}x_1 + \frac{b}{a+b}y_1\right)\right) \\
&\quad + \gamma^{-1}\left((a+b)\left(\frac{a}{a+b}x_1 + \frac{b}{a+b}y_1\right)\right) \\
&\geq \frac{a}{a+b}\left[\psi^{-1}((a+b)x_1) + \gamma^{-1}((a+b)x_2)\right] \\
&\quad + \frac{b}{a+b}\left[\psi^{-1}((a+b)y_1) + \gamma^{-1}((a+b)y_2)\right]
\end{aligned}$$

for all $x_1, x_2, y_1, y_2 > 0$. By (i) the function φ is decreasing and convex. Hence, using in turn: the monotonicity of φ, the convexity of φ and the above equivalent form of condition (iii), we get

$$\begin{aligned}
\varphi\left[\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_1 + by_1)\right] &\leq \varphi\left(\frac{a}{a+b}\left[\psi^{-1}((a+b)x_1) + \gamma^{-1}((a+b)x_2)\right]\right. \\
&\quad \left. + \frac{b}{a+b}\left[\psi^{-1}((a+b)y_1) + \gamma^{-1}((a+b)y_2)\right]\right) \\
&\leq \frac{a}{a+b}\varphi\left[\psi^{-1}((a+b)x_1) + \gamma^{-1}((a+b)x_2)\right] \\
&\quad + \frac{b}{a+b}\varphi\left[\psi^{-1}((a+b)y_1) + \gamma^{-1}((a+b)y_2)\right] \\
&\leq a\varphi\left[\psi^{-1}(x_1) + \gamma^{-1}(x_2)\right] + b\varphi\left[\psi^{-1}(y_1) + \gamma^{-1}(y_2)\right]
\end{aligned}$$

for all $x_1, x_2, y_1, y_2 > 0$.

Replacing here x_1, x_2, y_1, y_2 respectively by $\psi(x_1), \psi(x_2), \gamma(y_1), \gamma(y_2)$ and making again use of the decreasing monotonicity of φ, we obtain the required inequality (1.2). This completes the proof. □

Example. The functions $\varphi, \psi, \gamma : (0,\infty) \to (0,\infty)$ given by

$$\varphi(t) := \frac{1}{t}, \qquad \psi(t) := t^2, \qquad \gamma := e^t - 1, \quad t > 0,$$

are homeomorphisms of $(0,\infty)$ and, obviously, satisfy conditions (i) and (ii) of Proposition 1. Moreover, for arbitrary $a, b > 0$ we have

$$f(t) = \frac{t}{a+b}, \qquad g(t) = \sqrt{a+b}\,t, \qquad h(t) = \log\left((a+b)\left(e^t - 1\right) + 1\right), \quad t > 0.$$

If $a + b \geq 1$, then we have $h(t) \geq t$ for all $t > 0$, and, consequently, condition (iii) is satisfied.

Proposition 2. *Let $a > 0$ be arbitrarily fixed. If the bijections $\varphi, \psi, \gamma : (0,\infty) \to (0,\infty)$ such that*

(i) *φ is decreasing;*
(ii) *ψ and γ are increasing;*
(iii) *the functions*

$$f := \varphi^{-1} \circ (a\varphi), \qquad g := \psi^{-1} \circ (a\psi), \qquad h := \gamma^{-1} \circ (a\gamma)$$

satisfy the following Pexider type subadditivity condition

$$f(s+t) \leq g(s) + h(t), \quad s, t > 0,$$

then (sharp) inequality (1.2) holds for all $b > 0$ and $x_1, x_2, y_1, y_2 > 0$.

Proof. Take arbitrary $b, x_1, x_2, y_1, y_2 > 0$. Applying in turn: the decreasing monotonicity of φ^{-1}, condition (iii), the increasing monotonicity of ψ^{-1} and γ^{-1}, we obtain

$$\begin{aligned}\varphi^{-1}\left(a\varphi(x_1+y_1) + b\varphi(x_2+y_2)\right) &< \varphi^{-1}\left(a\varphi(x_1+y_1)\right)\\ &< \psi^{-1}\left(a\psi(x_1)\right) + \gamma^{-1}\left(a\gamma(y_1)\right)\\ &< \psi^{-1}\left(a\psi(x_1) + b\psi(x_2)\right) + \gamma^{-1}\left(a\gamma(y_1) + b\gamma(y_2)\right),\end{aligned}$$

which completes the proof. □

Remark 2. Let real numbers a, b be such that $0 < \min\{a, b) < 1 < a + b$. Take a measure space (Ω, Σ, μ) with $\Sigma = \{\emptyset, \Omega, A, B\}$ where $B = \Omega \setminus A$, such that $\mu(A) = a$ and $\mu(B) = b$. Obviously, we have $S_+(\Omega, \Sigma, \mu) = S_+(A, B)$ (cf. Remark 1). The bijections φ, ψ, γ given in Proposition 1 (or Proposition 2) with φ decreasing, need not be power functions, but they satisfy inequality (1.1). This proves that the increasing monotonicity of the function φ in Theorem 1 is essential.

The significance of the increasing monotonicity of the functions φ, ψ, γ in Theorem 1 explains also the following.

Proposition 3. *Let $a, b > 0$ be fixed. Then there does not exist a triple of decreasing bijections $\varphi, \psi, \gamma : (0,\infty) \to (0,\infty)$ such that inequality (1.2) holds true for all $x_1, x_2, y_1, y_2 > 0$.*

Proof. Suppose, for an indirect argument, that there exist decreasing bijections φ, ψ, γ of $(0,\infty)$ (so homeomorphisms) such that inequality (1.2) holds true. Of course, for all $y_1 > 0$, we have

$$\lim_{y_2 \to 0} \gamma^{-1}\left(a\gamma(y_1) + b\gamma(y_2)\right) = 0,$$

and, for all $x_2 > 0$,

$$\lim_{x_1 \to 0} \psi^{-1}\left(a\psi(x_1) + b\psi(x_2)\right) = 0.$$

Hence, letting y_2 tend to 0 in inequality (1.2), and making use of the continuity of the functions φ and φ^{-1}, we get

$$\varphi^{-1}(a\varphi(x_1+y_1) + b\varphi(x_2)) \leq \psi^{-1}(a\psi(x_1) + b\psi(x_2))$$

for all $x_1, x_2, y_1 > 0$, and letting here $x_1 \to 0$, we get

$$\varphi^{-1}(a\varphi(y_1) + b\varphi(x_2)) \leq 0, \quad y_1, x_2 > 0,$$

that is a contradiction. □

3. The Pexider type generalization of the Minkowski inequality in case (I)

We begin with some auxiliary notions and results.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be (cf. [9]) *geometrically convex* if

$$\varphi\left(s^r t^{1-r}\right) \leq [\varphi(s)]^r [\varphi(t)]^{1-r}, \quad s, t > 0, \ r \in (0, 1);$$

geometrically concave if the reversed inequality is satisfied, and *geometrically affine* if

$$\varphi\left(s^r t^{1-r}\right) = [\varphi(s)]^r [\varphi(t)]^{1-r}, \quad s, t > 0, \ r \in (0, 1).$$

Note the following properties of geometrically convex functions.

Properties.

(i) $\varphi : (0, \infty) \to (0, \infty)$ is geometrically convex (respectively geometrically concave, geometrically affine) iff $\log \circ \varphi \circ \exp$ is convex (respectively concave, affine) in $\mathbb{R}$.

(ii) If $\varphi : (0, \infty) \to (0, \infty)$ is geometrically convex, then φ'_- and φ'_+, the left and right derivatives of φ exist, and the functions

$$(0, \infty) \ni t \longmapsto \frac{\varphi'_-(t)}{\varphi(t)} t, \qquad (0, \infty) \ni t \longmapsto \frac{\varphi'_+(t)}{\varphi(t)} t$$

are increasing in $(0, \infty)$. Conversely, if φ is continuous, one-sided differentiable and one of the above functions is increasing, then φ is geometrically convex.

(iii) Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is bijective and increasing. Then φ is geometrically convex iff φ^{-1} is geometrically concave.

(iv) Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is bijective and decreasing. Then φ is geometrically convex iff φ^{-1} is geometrically convex.

(v) If $\varphi : (0, \infty) \to (0, \infty)$ is geometrically convex and the function $(0, \infty) \ni t \longrightarrow \frac{\varphi(t)}{t}$ is increasing, then φ is convex.

The proofs are easy. For instance, to show (v) observe that

$$\varphi'_+(t) = \left(\frac{\varphi'_+(t)}{\varphi(t)} t\right)\left(\frac{\varphi(t)}{t}\right),$$

and, in view of (ii), the function φ'_+ is increasing in $(0, \infty)$, as the product of positive increasing functions.

Each of the lemmas below generalizes the Mulholland result [6].

Lemma 1. *Suppose that $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective and such that*

(i) *φ is geometrically convex;*

(ii) *ψ'_- and γ'_- (or ψ'_+ and γ'_+) exist ψ'_- and γ'_- and the functions*

$$(0, \infty) \ni u \longmapsto \frac{\psi(u)}{u}, \qquad (0, \infty) \ni u \longmapsto \frac{\gamma(u)}{u}$$

are nondecreasing;

(iii) *the functions $\frac{\varphi}{\psi}$ and $\frac{\varphi}{\gamma}$ are nondecreasing.*

Then, for arbitrarily fixed $s, t > 0$, the function $F : (0, s] \times (0, t] \to (0, \infty)$ given by

$$F(u, v) := \frac{tu\psi^{-1}(s) + sv\gamma^{-1}(t)}{st\varphi\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)}, \quad 0 < u \le s, 0 < v \le t, \tag{3.1}$$

admits the minimum at the point (s, t), i.e.

$$0 < u \le s, \quad 0 < v \le t \Longrightarrow F(u, v) \ge F(s, t).$$

Proof. Suppose, for instance, that ψ and γ are left-side differentiable. For the simplicity of notations, by φ', ψ', γ' we denote the left-side derivatives $\varphi'_-, \psi'_-, \gamma'_-$, respectively.

By (ii) the functions ψ and γ are increasing. This assumption implies also that $\psi' > 0$ and $\gamma' > 0$. In view of (ii), the left derivative $\varphi' = \varphi'_-$ exists in $(0, \infty)$. From (iii) we have

$$\frac{\psi'(u)}{\psi(u)} \le \frac{\varphi'(u)}{\varphi(u)}, \qquad \frac{\gamma'(u)}{\gamma(u)} \le \frac{\varphi'(u)}{\varphi(u)}, \quad u > 0. \tag{3.2}$$

Each of these inequalities together with inequalities $\psi' > 0, \gamma' > 0$ implies that $\varphi' > 0$. It follows that φ, ψ, γ are increasing homeomorphisms of $(0, \infty)$. By the chain rule (for the left derivative), at every point of the set $(0, s] \times (0, t]$, there exists the left partial derivative of F and

$$\frac{\partial(\log \circ F)}{\partial u}(u, v) = \frac{t\psi^{-1}(s)}{tu\psi^{-1}(u) + sv\gamma^{-1}(v)} - \frac{\varphi'\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)}{\varphi\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)} \frac{1}{\psi'\left(\psi^{-1}(u)\right)},$$

$$\frac{\partial(\log \circ F)}{\partial v}(u, v) = \frac{s\gamma^{-1}(t)}{tu\psi^{-1}(u) + sv\gamma^{-1}(v)} - \frac{\varphi'\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)}{\varphi\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)} \frac{1}{\gamma'\left(\gamma^{-1}(v)\right)}.$$

Making use of the geometrical convexity of φ, property (ii) and (3.2), we get

$$\begin{aligned}\frac{\varphi'\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)}{\varphi\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)}\frac{1}{\psi'\left(\psi^{-1}(u)\right)} &= \frac{\varphi'\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)}{\varphi\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)}\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)\\ &\quad\times\frac{1}{\psi^{-1}(u)+\gamma^{-1}(v)}\frac{1}{\psi'\left(\psi^{-1}(u)\right)}\\ &\ge \frac{\varphi'\left(\psi^{-1}(u)\right)}{\varphi\left(\psi^{-1}(u)\right)}\psi^{-1}(u)\frac{1}{\psi^{-1}(u)+\gamma^{-1}(v)}\frac{1}{\psi'\left(\psi^{-1}(u)\right)}\\ &\ge \frac{\psi'\left(\psi^{-1}(u)\right)}{\psi\left(\psi^{-1}(u)\right)}\psi^{-1}(u)\frac{1}{\psi^{-1}(u)+\gamma^{-1}(v)}\frac{1}{\psi'\left(\psi^{-1}(u)\right)}\\ &= \frac{\psi^{-1}(u)}{u\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)},\end{aligned}$$

whence

$$\frac{\partial(\log\circ F)}{\partial u}(u,v)\le\frac{t\psi^{-1}(s)}{tu\psi^{-1}(u)+sv\gamma^{-1}(v)}-\frac{\psi^{-1}(u)}{u\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)},$$

for all $0<u\le s, 0<v\le t$.

In the same way we can show that

$$\frac{\partial(\log\circ F)}{\partial v}(u,v)\le\frac{s\gamma^{-1}(t)}{tu\psi^{-1}(u)+sv\gamma^{-1}(v)}-\frac{\gamma^{-1}(v)}{v\left(\psi^{-1}(u)+\gamma^{-1}(v)\right)},$$

for all $0<u\le s, 0<v\le t$.

Now consider the following three types of curves

$$\text{(I)}\begin{cases}u(\tau)=k\tau\\ v(\tau)=m\tau\end{cases}\quad (k,m\ge 0);\qquad \text{(II)}\begin{cases}u(\tau)=s\\ v(\tau)=\tau;\end{cases}\qquad \text{(III)}\begin{cases}u(\tau)=\tau\\ v(\tau)=t\end{cases}$$

passing inside the set $(0,s]\times(0,t]$.

Let us note that to prove our lemma it is sufficient to show that in each of these three cases the function

$$h(\tau):=(\log\circ F)\left(u(\tau),v(\tau)\right)$$

is nonincreasing.

Put $h':=h'_-$. For an arbitrary regular curve $(0,T]\ni\tau\longmapsto(u(\tau),v(\tau))$ passing inside the set $(0,s]\times(0,t]$, applying the above inequalities, performing simple calculations, we obtain, for all $\tau\in(0,T]$

$$\begin{aligned}h'(\tau) &= \frac{\partial(\log\circ F)}{\partial u}\left(u(\tau),v(\tau)\right)u'(\tau)+\frac{\partial(\log\circ F)}{\partial v}\left(u(\tau),v(\tau)\right)v'(\tau)\\ &\le\left[\frac{t\psi^{-1}(s)}{tu(\tau)\,\psi^{-1}(u(\tau))+sv(\tau)\,\gamma^{-1}(v(\tau))}-\frac{\psi^{-1}(u(\tau))}{u(\tau)\left(\psi^{-1}(u(\tau))+\gamma^{-1}(v(\tau))\right)}\right]u'(\tau)\\ &\quad+\left[\frac{s\gamma^{-1}(t)}{tu(\tau)\,\psi^{-1}(u(\tau))+sv(\tau)\,\gamma^{-1}(v(\tau))}-\frac{\gamma^{-1}(v(\tau))}{v(\tau)\left(\psi^{-1}(u(\tau))+\gamma^{-1}(v(\tau))\right)}\right]v'(\tau).\end{aligned}$$

Substituting here an arbitrary curve of type (I) we get $h'(\tau)\le 0$ for all $\tau\in(0,T]$.

Substituting here an arbitrary curve of type (II), and taking into account that $u'(\tau)=0$, $v'(\tau)=1$, we hence get, for all $\tau\in(0,t]$,

$$\begin{aligned}h'(\tau) &\le \frac{s\gamma^{-1}(t)}{tu(\tau)\,\psi^{-1}(u(\tau))+sv(\tau)\,\gamma^{-1}(v(\tau))}-\frac{\gamma^{-1}(v(\tau))}{v(\tau)\left(\psi^{-1}(u(\tau))+\gamma^{-1}(v(\tau))\right)}\\ &= \frac{s\gamma^{-1}(t)}{ts\psi^{-1}(s)+s\tau\gamma^{-1}(\tau)}-\frac{\gamma^{-1}(\tau)}{\tau\left(\psi^{-1}(s)+\gamma^{-1}(\tau)\right)}\\ &= \frac{\psi^{-1}(s)\left[\gamma^{-1}(t)\tau-\gamma^{-1}(\tau)t\right]+\tau\gamma^{-1}(\tau)\left[\gamma^{-1}(t)-\gamma^{-1}(\tau)\right]}{\left[ts\psi^{-1}(s)+s\tau\gamma^{-1}(\tau)\right]\left[\tau\left(\psi^{-1}(s)+\gamma^{-1}(\tau)\right)\right]}\\ &= \frac{\psi^{-1}(s)t\tau\left[\frac{\gamma^{-1}(t)}{t}-\frac{\gamma^{-1}(\tau)}{\tau}\right]+\tau\gamma^{-1}(\tau)\left[\gamma^{-1}(t)-\gamma^{-1}(\tau)\right]}{\left[ts\psi^{-1}(s)+s\tau\gamma^{-1}(\tau)\right]\left[\tau\left(\psi^{-1}(s)+\gamma^{-1}(\tau)\right)\right]}.\end{aligned}$$

Since γ^{-1} is increasing and, by assumption, the function $\tau \longmapsto \frac{\gamma^{-1}(\tau)}{\tau}$ is increasing, the function being the restriction of the right-hand side of above inequality to the interval $(0, t]$ attains its maximum equal 0 at the point $\tau = t$. This proves that $h'(\tau) \le 0$ for all $\tau \in (0, t]$.

We omit a similar reasoning in the case when the curve is of type (III). This completes the proof. □

Note that the same argument permits to prove the following n-dimensional counterpart of Lemma 1:

Lemma 2. *Let $n \in \mathbb{N}$, $n \ge 2$ be fixed. Suppose that $\varphi, \psi_1, \ldots, \psi_n : (0, \infty) \to (0, \infty)$ are bijective and such that*

(i) *φ is geometrically convex;*

(ii) *for each $i \in \{1, \ldots, n\}$, the function ψ_i is left-differentiable (or for each $i \in \{1, \ldots, n\}$, the function ψ_i is right-differentiable) and the function*

$$(0, \infty) \ni t \longmapsto \frac{\psi_i(t)}{t}$$

is nondecreasing;

(iii) *for each $i \in \{1, \ldots, n\}$, the function $\frac{\varphi}{\psi_i}$ is nondecreasing.*

Then, for arbitrary real numbers $s_1, \ldots, s_n > 0$, the function $F : (0, s_1] \times \cdots \times (0, s_n] \to (0, \infty)$ given by

$$F(x_1, \ldots, x_n) := \frac{s_1^{-1}x_1\psi_1^{-1}(s_1) + \cdots + s_n^{-1}x_n\psi_1^{-1}(s_n)}{\varphi\left(\psi_1^{-1}(x_1) + \cdots + \psi_1^{-1}(x_n)\right)}$$

admits the minimum at the point $(s_1, \ldots, s_n)$, i.e.

$$0 < x_1 \le s_1, \ldots, 0 < x_n \le s_n \Longrightarrow F(x_1, \ldots, x_n) \ge F(s_1, \ldots, s_n).$$

Similarly as Lemma 1 we can prove the following

Lemma 3. *Suppose that $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective and such that*

(i) *φ is geometrically concave;*

(ii) *ψ'_- and γ'_- (or ψ'_+ and γ'_+) exist φ'_- and γ'_- and the functions*

$$(0, \infty) \ni u \longrightarrow \frac{\psi(u)}{u}, \qquad (0, \infty) \ni u \longrightarrow \frac{\gamma(u)}{u}$$

are nonincreasing;

(iii) *the functions $\frac{\varphi}{\psi}$ and $\frac{\varphi}{\gamma}$ are nonincreasing.*

Then, for arbitrarily fixed $s, t > 0$, the function $F : (0, s] \times (0, t] \to (0, \infty)$ given by

$$F(u, v) := \frac{tu\psi^{-1}(s) + sv\gamma^{-1}(t)}{st\varphi\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)}, \quad 0 < u \le s,\ 0 < v \le t,$$

admits the maximum at the point (s, t).

Let us also note the nth dimensional counterpart of Lemma 3.

Lemma 4. *Let $n \in \mathbb{N}$, $n \ge 2$ be fixed. Suppose that $\varphi, \psi_1, \ldots, \psi_n : (0, \infty) \to (0, \infty)$ are bijective and such that*

(i) *φ is geometrically concave;*

(ii) *for each $i \in \{1, \ldots, n\}$, the function ψ_i is left-differentiable (or for each $i \in \{1, \ldots, n\}$, the function ψ_i is right-differentiable) and the function*

$$(0, \infty) \ni t \longmapsto \frac{\psi_i(t)}{t}$$

is nonincreasing;

(iii) *for each $i \in \{1, \ldots, n\}$, the function $\frac{\varphi}{\psi_i}$ is nonincreasing.*

Then, for arbitrary real numbers $s_1, \ldots, s_n > 0$, the function $F : (0, s_1] \times \cdots \times (0, s_n] \to (0, \infty)$ given by

$$F(x_1, \ldots, x_n) := \frac{s_1^{-1}x_1\psi_1^{-1}(s_1) + \cdots + s_n^{-1}x_n\psi_1^{-1}(s_n)}{\varphi\left(\psi_1^{-1}(x_1) + \cdots + \psi_1^{-1}(x_n)\right)}$$

admits the maximum at the point $(s_1, \ldots, s_n)$.

Now we can prove the existence of the broad classes of triples (φ, ψ, γ) of non-power functions satisfying inequality (P) in the case when the underlying measure space (Ω, Σ, μ) satisfies the condition

(I) *for every $A \in \Sigma$, we have $\mu(A) = 0$ or $\mu(A) \ge 1$.*

Theorem 2. *Let (Ω, Σ, μ) be a measure space with $\Sigma = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for some $A \subset \Omega$ such that $\mu(A)$, $\mu(\Omega \setminus A)$ are positive real numbers, and*

$$\min\{\mu(A), \mu(\Omega \setminus A)\} \geq 1.$$

Suppose that $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective and such that
(i) *φ is geometrically convex;*
(ii) *φ'_- and γ'_- (or φ'_+ and γ'_+) exist φ'_- and γ'_- and the functions*

$$(0, \infty) \ni u \longrightarrow \frac{\varphi(u)}{u}, \qquad (0, \infty) \ni u \longrightarrow \frac{\gamma(u)}{u}$$

are nondecreasing;
(iii) *the functions $\frac{\varphi}{\psi}$ and $\frac{\varphi}{\gamma}$ are nondecreasing;*
(iv) *for each $r \in \mu(\Sigma) \setminus \{0\}$, the functions*

$$f_r := \varphi^{-1} \circ (r\varphi), \qquad g_r := \psi^{-1} \circ (r\psi), \qquad h_r := \gamma^{-1} \circ (r\gamma)$$

satisfy the "Pexiderized" subadditivity condition:

$$f_r(u + v) \leq g_r(u) + h_r(v), \quad u, v > 0.$$

Then

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu).$$

Proof. Take arbitrary $\mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu)$. Then such that

$$\mathbf{x} = x_1\chi_A + x_2\chi_B, \qquad \mathbf{y} = y_1\chi_A + y_2\chi_B,$$

for some $x_1, x_2, y_1, y_2 > 0$, where $B := \Omega \setminus A$. Put

$$s := x_1 + x_2, \qquad t := y_1 + y_2,$$

and define $F : (0, s] \times (0, t] \to (0, \infty)$ by (3.1). According to Lemma 1 we have $F(x_1, x_2) \geq F(s, t)$, i.e.

$$\frac{tx_1\psi^{-1}(s) + sy_1\gamma^{-1}(t)}{st\varphi\left(\psi^{-1}(x_1) + \gamma^{-1}(y_1)\right)} \geq \frac{\psi^{-1}(s) + \gamma^{-1}(t)}{\varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)}$$

and

$$\frac{tx_2\psi^{-1}(s) + sy_2\gamma^{-1}(t)}{st\varphi\left(\psi^{-1}(x_2) + \gamma^{-1}(y_2)\right)} \geq \frac{\psi^{-1}(s) + \gamma^{-1}(t)}{\varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)}.$$

Multiplying the first of the above inequalities by $\varphi\left(\psi^{-1}(x_1) + \gamma^{-1}(y_1)\right)$, the second one by $\varphi\left(\psi^{-1}(x_2) + \gamma^{-1}(y_2)\right)$, adding the respective sides of the resulting inequalities, we get

$$\frac{t(x_1 + x_2)\psi^{-1}(s) + s(y_1 + y_2)\gamma^{-1}(t)}{st} \geq \frac{\psi^{-1}(s) + \gamma^{-1}(t)}{\varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)}\left[\varphi\left(\psi^{-1}(x_1) + \gamma^{-1}(y_1)\right) + \varphi\left(\psi^{-1}(x_2) + \gamma^{-1}(y_2)\right)\right]$$

whence, making use of the definition of the numbers s and t, we obtain

$$\varphi\left(\psi^{-1}(x_1 + y_1) + \gamma^{-1}(x_2 + y_2)\right) \geq \varphi\left(\psi^{-1}(x_1) + \gamma^{-1}(x_2)\right) + \varphi\left(\psi^{-1}(y_1) + \gamma^{-1}(y_2)\right)$$

for arbitrary $x_1, x_2, y_1, y_2 > 0$.

Put

$$a := \mu(A), \qquad b := \mu(\Omega \setminus A).$$

Let $r \in \{a, b\}$. From the "Pexiderized" subadditivity assumed in (iv) we have

$$\varphi^{-1} \circ (r\varphi)(u + v) \leq \psi^{-1} \circ (r\psi)(u) + \gamma^{-1} \circ (r\gamma)(v), \quad u, v > 0,$$
$$\varphi^{-1}(r(\varphi(u + v))) \leq \psi^{-1}(r\psi(u)) + \gamma^{-1}(r\gamma(v)), \quad r \in \{a, b\}, \ u, v > 0.$$

Replacing u by $\psi^{-1}(u)$ and v by $\gamma^{-1}(v)$, we get

$$\varphi^{-1}\left(r\left(\varphi\left(\psi^{-1}(u) + \gamma^{-1}(v)\right)\right)\right) \leq \psi^{-1}(ru) + \gamma^{-1}(rv), \quad r \in \{a, b\}, \ u, v > 0,$$

whence, taking into account that φ is increasing (that is a consequence of (ii) and (iii)), we obtain

$$\varphi\left(\psi^{-1}(ru)+\gamma^{-1}(rv)\right)\geq r\varphi\left(\psi^{-1}(u)+\gamma^{-1}(v)\right),\quad r\in\{a,b\},\ u,v>0.$$

Making use of the previous inequality, we hence get, for all $x_1, x_2, y_1, y_2 > 0$,

$$\begin{aligned}\varphi\left(\psi^{-1}(ax_1+by_1)+\gamma^{-1}(ax_2+by_2)\right) &\geq \varphi\left(\psi^{-1}(ax_1)+\gamma^{-1}(ax_2)\right)+\varphi\left(\psi^{-1}(by_1)+\gamma^{-1}(by_2)\right)\\ &\geq a\varphi\left(\psi^{-1}(x_1)+\gamma^{-1}(x_2)\right)+b\varphi\left(\psi^{-1}(y_1)+\gamma^{-1}(y_2)\right).\end{aligned}$$

Since $x_1, x_2, y_1, y_2 > 0$ can be taken arbitrarily, replacing here x_i by $\psi(x_i)$, y_i by $\gamma(y_i)$ $(i = 1, 2)$, and making use of the increasing monotonicity of the function φ^{-1}, we obtain

$$\begin{aligned}\mathbb{P}_\psi(\mathbf{x})+\mathbb{P}_\gamma(\mathbf{y}) &= \psi^{-1}(a\psi(x_1)+b\psi(x_2))+\gamma^{-1}(a\gamma(y_1)+b\gamma(y_2))\\ &\geq \varphi^{-1}(a\varphi(x_1+y_1)+b\varphi(x_2+y_2))=\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y}),\end{aligned}$$

which completes the proof. □

Remark 3. The assumption (iv) of Theorem 2 is equivalent to the condition that for each $r \in \{\mu(A), \mu(\Omega\setminus A)\}$,

$$r\varphi(\psi^{-1}(u)+\gamma^{-1}(v))\leq\varphi(\psi^{-1}(ru)+\gamma^{-1}(rv)),\quad u,v>0.$$

This assumption is satisfied if $\mu(A) = \mu(\Omega \setminus A) = 1$.

Arguing similarly as in the proof of Theorem 2 and applying Lemma 2, we can prove the following

Theorem 3. *Let $n \in \mathbb{N}$, $n \geq 2$ be fixed and (Ω, Σ, μ) be a measure space such that $\Sigma = \{\emptyset, A_1, \ldots, A_n\}$ for some pairwise disjoint sets $A_i \subset \Omega$, $i = 1, \ldots, n$, of positive finite measures and such that*

$$\min\{\mu(A_i): i=1,\ldots,n\}\geq 1.$$

Suppose that $\varphi, \psi_1, \ldots, \psi_n : (0,\infty) \to (0,\infty)$ are bijective and such that

(i) *φ is geometrically convex;*

(ii) *for each $i \in \{1,\ldots,n\}$, the function ψ_i is left-differentiable (or for each $i \in \{1,\ldots,n\}$, the function ψ_i is right-differentiable) and the function*

$$(0,\infty)\ni t\longmapsto\frac{\psi_i(t)}{t}$$

is nondecreasing;

(iii) *for each $i \in \{1,\ldots,n\}$, the function $\frac{\varphi}{\psi_i}$ is nondecreasing;*

(iv) *for each $r \in \mu(\Sigma)\setminus\{0\}$, the functions*

$$f_r := \varphi^{-1}\circ(r\varphi);\qquad g_{i,r} := \psi_i^{-1}\circ(r\psi_i),\quad i=1,\ldots,n,$$

satisfy the "Pexiderized" subadditivity condition:

$$f_r(u_1+\cdots+u_n)\leq g_{1,r}(u_1)+\cdots+g_{n,r}(u_n),\quad u_1,\ldots,u_n>0.$$

Then

$$\mathbb{P}_\varphi(\mathbf{x}_1+\cdots+\mathbf{x}_n)\leq\mathbb{P}_\varphi(\mathbf{x}_1)+\cdots+\mathbb{P}_\varphi(\mathbf{x}_n),\quad \mathbf{x}_1,\ldots,\mathbf{x}_n\in S_+(\Omega,\Sigma,\mu).$$

Remark 4. The assumption (iv) of the above theorem is equivalent to the inequality

$$r\varphi(\psi^{-1}(u)+\gamma^{-1}(v))\leq\varphi(\psi^{-1}(ru)+\gamma^{-1}(rv)),\quad r\in\mu(\Sigma)\setminus\{0\},\ u,v>0,$$

and, obviously, it is satisfied in the case when $\mu(A_1) = \cdots = \mu(A_n) = 1$.

At the end of this section we prove the following

Proposition 4. *Let (Ω, Σ, μ) be a measure space with $\Sigma = \{\emptyset, \Omega, A, \Omega\setminus A\}$ for some $A \subset \Omega$ such that $\mu(A), \mu(\Omega\setminus A)$ are positive real numbers, and*

$$\min\{\mu(A),\mu(\Omega\setminus A)\}\geq 1.$$

If $\varphi, \psi, \gamma : (0,\infty) \to (0,\infty)$ are homeomorphisms such that

(i) *φ is decreasing;*

(ii) *$\psi = \gamma$ is increasing;*

then

$$\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y})<\mathbb{P}_\psi(\mathbf{x})+\mathbb{P}_\gamma(\mathbf{y}),\quad \mathbf{x},\mathbf{y}\in S_+(\Omega,\Sigma,\mu),$$

(that is sharp inequality is satisfied).

Proof. Take arbitrary $\mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu)$. Then there exist $x_1, x_2, y_1, y_2 > 0$ such that

$$\mathbf{x} = x_1\chi_A + x_2\chi_B, \qquad \mathbf{y} = y_1\chi_A + y_2\chi_B,$$

where $B := \Omega \setminus A$. Put $a := \mu(A)$, $b := \mu(B)$. Since $\psi^{-1} = \gamma^{-1}$ is increasing, we have

$$\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2) > \psi^{-1}(x_1) + \gamma^{-1}(x_2).$$

Hence, as φ is decreasing,

$$\varphi\left(\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2)\right) < \varphi\left(\psi^{-1}(x_1) + \gamma^{-1}(x_2)\right)$$

and, consequently,

$$\varphi\left(\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2)\right) < a\varphi\left(\psi^{-1}(x_1) + \gamma^{-1}(x_2)\right) + b\varphi\left(\psi^{-1}(y_1) + \gamma^{-1}(y_2)\right).$$

Since this inequality holds true for all $x_1, x_2, y_1, y_2 > 0$, replacing here x_1, x_2, y_1, y_2, respectively by $\psi(x_1)$, $\psi(x_2)$, $\gamma(y_1)$, $\gamma(y_2)$ and making again use of the decreasing monotonicity of φ, we obtain

$$\begin{aligned}\mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}) &= \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)) \\ &\geq \varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) = \mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}),\end{aligned}$$

which completes the proof. □

4. The Pexider type generalization of the Minkowski inequality in case (II)

In this section we consider the case when the underlying measure space (Ω, Σ, μ) satisfies the condition

(II) *for every $A \in \Sigma$, we have $\mu(A) \leq 1$ or $\mu(A) = \infty$.*

First we consider the case $\mu(\Omega) = 1$. We begin with

Theorem 4. *Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) = 1$, $\Sigma = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for some $A \subset \Omega$ and such that $0 < \mu(A) < 1$. Suppose that $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ are bijective functions.*

(i) *If φ is increasing, then inequality (P):*

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu),$$

holds true if, and only if, the function $\Phi : (0, \infty)^2 \to (0, \infty)$ defined by

$$\Phi(s, t) := \varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right), \quad s, t > 0, \tag{4.1}$$

is concave.

(ii) *If φ is decreasing, then inequality (P) holds true if, and only if, the function Φ is $\mu(A)$-convex, that iff*

$$\Phi(as_1 + (1 - a)s_2, at_1 + (1 - a)t_2) \leq a\Phi(s_1, t_1) + (1 - a)\Phi(s_2, t_2), \quad s_1, s_2, t_1, t_2 > 0,$$

where $a := \mu(A)$. If, moreover, ψ and γ are continuous, then Φ is convex.

Proof. (i) Put $a := \mu(A)$, $b := \mu(\Omega \setminus A) = 1 - a$. By Remark 1, the inequality (P) is equivalent to the inequality

$$\varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) \leq \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2))$$

for all $x_1, y_1, x_2, y_2 > 0$. Replacing here x_i by $\psi^{-1}(x_i)$, y_i by $\gamma^{-1}(y_i)$ $(i = 1, 2)$, by the increasing monotonicity of φ, we obtain the equivalent inequality

$$a\varphi(\psi^{-1}(x_1) + \gamma^{-1}(y_1)) + b\varphi(\psi^{-1}(x_2) + \gamma^{-1}(y_2)) \leq \varphi\left(\psi^{-1}(ax_1 + bx_2) + \gamma^{-1}(ay_1 + by_2)\right)$$

for all $x_1, y_1, x_2, y_2 > 0$, which says that the function Φ is a-concave. By the Daróczy–Páles identity [10], the function Φ is Jensen concave. Since Φ is bounded from below, by Bernstein–Doetsch theorem (cf. [7]), it is continuous, and, consequently, Φ is concave.

(ii) We omit an analogous argument. □

Remark 5. In the case (ii) when φ is decreasing, in view of Proposition 3, at least one of the functions ψ and γ must be increasing.

In a similar way one can prove

Theorem 5. *Let $n \in \mathbb{N}$, $n \geq 2$ be fixed and (Ω, Σ, μ) be a measure space such that $\mu(\Omega) = 1$, $\Sigma = \{\emptyset, A_1, \ldots, A_n\}$ for some pairwise disjoint sets $A_i \subset \Omega$, $i = 1, \ldots, n$, of positive measures. Suppose that $\varphi, \psi_1, \ldots, \psi_n : (0, \infty) \to (0, \infty)$ are bijective.*

(i) *If* φ *is increasing, then*

$$\mathbb{P}_{\varphi}(\mathbf{x}_1 + \cdots + \mathbf{x}_n) \le \mathbb{P}_{\psi_1}(\mathbf{x}_1) + \cdots + \mathbb{P}_{\psi_n}(\mathbf{x}_n), \quad \mathbf{x}_1, \ldots, \mathbf{x}_n \in S_+(\Omega, \Sigma, \mu), \tag{4.2}$$

if, and only if, the function $\Phi : (0, \infty)^n \to (0, \infty)$ *defined by*

$$\Phi(t_1, \ldots, t_n) := \varphi\left(\psi_1^{-1}(t_1) + \cdots + \psi_n^{-1}(t_n)\right), \quad t_1, \ldots, t_n > 0, \tag{4.3}$$

is concave.

(ii) *if* φ *is decreasing and* $\psi_1, \ldots, \psi_n$ *are continuous, then inequality* (4.2) *f, and only if, the function* (4.3) *is convex.*

Lemma 5. *Let* $\varphi, \psi, \gamma : (0, \infty) \to (0, \infty)$ *be bijective, twice differentiable, such that* $\varphi'\psi'\gamma'\varphi''\psi''\gamma'' \neq 0$ *in* $(0, \infty)$.

(i) *If* $\varphi' > 0$ *then* Φ *defined by* (4.1) *is concave if, and only if,*

$$\frac{\psi''(s)}{\psi'(s)} \le \frac{\varphi''(s+t)}{\varphi'(s+t)}, \qquad \frac{\gamma''(t)}{\gamma'(t)} \le \frac{\varphi''(s+t)}{\varphi'(s+t)}, \quad s, t > 0, \tag{4.4}$$

and

$$\frac{\varphi''(s+t)}{[\varphi'(s+t)]^3}\frac{\psi''(s)}{[\psi'(s)]^3}\frac{\gamma''(s)}{[\gamma'(s)]^3}\left(\frac{\varphi'(s+t)}{\varphi''(s+t)} - \frac{\psi'(s)}{\psi''(s)} - \frac{\gamma'(t)}{\gamma''(t)}\right) \ge 0, \quad s, t > 0. \tag{4.5}$$

(ii) *If* $\varphi' < 0$ *then* Φ *defined by* (4.1) *is concave if, and only if, inequality* (4.5) *and the reversed inequalities* (4.4) *are fulfilled.*

Proof. For arbitrarily fixed $s, t > 0$ and admissible $u, v \in \mathbb{R}$ define a function of a single variable

$$G(\tau) := \Phi(s + u\tau, t + v\tau) = \varphi\left(\psi^{-1}(s + u\tau) + \gamma^{-1}(t + v\tau)\right).$$

The function Φ is concave if, and only if, for all $s, t > 0$ and $u, v \in \mathbb{R}$ we have $G''(0) \le 0$ (cf. [11, pp. 79–81]). Put $f := \varphi^{-1}$, $g := \psi^{-1}$, $h := \gamma^{-1}$. From the definition of G we have

$$f(G(\tau)) = g(s + u\tau) + h(t + v\tau)$$

for τ in a neighborhood of 0. Hence we obtain

$$f'(G(0))\,G'(0) = g'(s)u + h'(t)v$$

and

$$f''(G(0))\left[G'(0)\right]^2 + f'(G(0))\,G''(0) = g''(s)u^2 + h''(t)v^2.$$

Eliminating $G'(0)$ from these two equalities, we get

$$\left[f'(G(0))\right]^3 G''(0) = Au^2 - 2Buv + Cv^2, \tag{4.6}$$

where

$$A := g''(s)\left[f'(G(0))\right]^2 - f''(G(0))\left[g'(s)\right]^2,$$
$$B := f''(G(0))\,g'(s)\,h'(t),$$
$$C := h''(t)\left[f'(G(0))\right]^2 - f''(G(0))\left[h'(t)\right]^2.$$

If φ' is positive, then $f'(G(0)) > 0$. From (4.6) we conclude that $G''(0) \le 0$ if

$$A \le 0, \qquad C \le 0, \qquad AC \ge B^2.$$

Note that

$$G(0) = \varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right), \qquad f' \circ \varphi = \frac{1}{\varphi'}, \qquad f'' \circ \varphi = -\frac{\varphi''}{(\varphi')^3},$$

$$g' \circ \psi = \frac{1}{\psi'}, \qquad g'' \circ \psi = -\frac{\psi''}{(\psi')^3}, \qquad h' \circ \gamma = \frac{1}{\gamma'}, \qquad h'' \circ \gamma = -\frac{\gamma''}{(\gamma')^3}.$$

It follows that, for all $s, t > 0$,

$$f'(G(0)) = f'\left(\varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)\right) = \frac{1}{\varphi'\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)},$$

$$f''(G(0)) = f''\left(\varphi\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)\right) = -\frac{\varphi''\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)}{\left[\varphi'\left(\psi^{-1}(s) + \gamma^{-1}(t)\right)\right]^3},$$

$$g'(s)=\frac{1}{\psi'\left(\psi^{-1}(s)\right)},\qquad g''(s)=-\frac{\psi''\left(\psi^{-1}(s)\right)}{\left[\psi'\left(\left(\psi^{-1}(s)\right)\right)\right]^3},$$

$$h'(t)=\frac{1}{\gamma'\left(\gamma^{-1}(t)\right)},\qquad h''(t)=-\frac{\delta''\left(\gamma^{-1}(t)\right)}{\left[\gamma'\left(\left(\gamma^{-1}(t)\right)\right)\right]^3}.$$

Hence, by the definitions of A, B, C, and replacing, for the simplicity of notations, s by $\psi(s)$, t by $\gamma(t)$, we obtain

$$A\le 0\iff \frac{\varphi''(s+t)}{\varphi'(s+t)}\le\frac{\psi''(s)}{\psi'(s)},\qquad C\le 0\iff \frac{\varphi''(s+t)}{\varphi'(s+t)}\le\frac{\gamma''(t)}{\gamma'(t)},$$

and

$$AC\ge B^2\iff \frac{\varphi''(s+t)}{[\varphi'(s+t)]^3}\frac{\psi''(s)}{[\psi'(s)]^3}\frac{\gamma''(s)}{[\gamma'(s)]^3}\left(\frac{\varphi'(s+t)}{\varphi''(s+t)}-\frac{\psi'(s)}{\psi''(s)}-\frac{\gamma'(t)}{\gamma''(t)}\right)\ge 0$$

for all $s, t > 0$, which completes the proof of the first part of the lemma.

Since the proof of the second part is analogous we omit it. □

From this lemma we obtain the following

Proposition 5. *Let $\varphi,\psi,\gamma:(0,\infty)\to(0,\infty)$ be bijective, twice differentiable with positive first and second derivatives in $(0,\infty)$. If*

$$\frac{\psi'(s)}{\psi''(s)}+\frac{\gamma'(t)}{\gamma''(t)}\le\frac{\varphi'(s+t)}{\varphi''(s+t)},\quad s,t>0, \tag{4.7}$$

then Φ defined by (4.1) is concave.

Proof. Since $\frac{\psi'}{\psi''},\frac{\gamma'}{\gamma''},\frac{\varphi'}{\varphi''}$ are positive, (4.7) implies reversed inequalities (4.4) and (4.5) in the above lemma. □

Remark 6. It is easy to formulate and prove the counterpart of the above proposition ensuring the concavity (convexity) of the function Φ defined by (4.3).

Theorem 6. *Let (Ω,Σ,μ) be a measure space such that $\mu(\Omega)\le 1$, $\Sigma=\{\emptyset,\Omega,A,\Omega\setminus A\}$ for some $A\subset\Omega$ and such that $0<\mu(A)<1$. Suppose that $\varphi,\psi,\gamma:(0,\infty)\to(0,\infty)$ are increasing bijective functions.*

If φ is increasing and the function $\Phi:(0,\infty)^2\to(0,\infty)$ defined by

$$\Phi(s,t):=\varphi\left(\psi^{-1}(s)+\gamma^{-1}(t)\right),\quad s,t>0,$$

is concave, then inequality (P) holds true.

Proof. The bijectivity and increasing monotonicity of the functions φ,ψ,γ imply that $\varphi(0+)=\psi(0+)=\gamma(0+)=0$. Therefore we may assume that $\varphi(0)=\psi(0)=\gamma(0)=0$. Then the function Φ_0

$$\Phi_0(s,t):=\varphi\left(\psi^{-1}(s)+\gamma^{-1}(t)\right),\quad s,t\ge 0$$

is well defined on $\mathbb{R}_+^2$. Since Φ_0 is continuous and $\Phi:=\Phi_0\big|_{(0,\infty)^2}$ is concave, it is obvious that Φ_0 is concave in $\mathbb{R}_+^2$, and $\Phi_0(0,0)=0$.

Take arbitrary $\mathbf{x},\mathbf{y}\in S_+(\Omega,\Sigma,\mu)$. Then there exist $x_1,x_2,y_1,y_2>0$ such that

$$\mathbf{x}=x_1\chi_A+x_2\chi_B,\qquad \mathbf{y}=y_1\chi_A+y_2\chi_B,$$

where $B:=\Omega\setminus A$. Putting $a:=\mu(A)$, $b:=\mu(B)$, by the concavity of Φ_0 we have

$$\begin{aligned}&a\varphi\left(\psi^{-1}(u_1)+\gamma^{-1}(v_1)\right)+b\varphi\left(\psi^{-1}(u_2)+\gamma^{-1}(v_2)\right)+(1-a-b)\varphi(\psi^{-1}(0)+\gamma^{-1}(0))\\&\quad\le\varphi(\psi^{-1}(au_1+bu_2+(1-a-b)0))+\gamma^{-1}(av_1+bv_2+(1-a-b)0),\end{aligned}$$

whence, for all $u_1,u_2,v_1,v_2>0$,

$$a\varphi(\psi^{-1}(u_1)+\gamma^{-1}(v_1))+b\varphi(\psi^{-1}(u_2)+\gamma^{-1}(v_2))\le\varphi\left(\psi^{-1}(au_1+bu_2)+\gamma^{-1}(av_1+bv_2)\right).$$

Setting here $u_1:=\varphi(x_1)$, $u_2:=\varphi(x_2)$, $v_1:=\varphi(y_1)$, $v_2:=\varphi(y_2)$, and making use of the increasing monotonicity of φ, we obtain

$$\varphi^{-1}(a\varphi(x_1+y_1)+b\varphi(x_2+y_2))\le\psi^{-1}(a\psi(x_1)+b\psi(x_2))+\gamma^{-1}(a\gamma(y_1)+b\gamma(y_2)),$$

that is

$$\mathbb{P}_\varphi(\mathbf{x}+\mathbf{y})\le\mathbb{P}_\varphi(\mathbf{x})+\mathbb{P}_\varphi(\mathbf{y}),$$

which completes the proof. □

Remark 7. Applying similar reasoning, one can easily get the suitable version of this result in the case when φ is decreasing and the counterparts of both results in the case when Σ is a finite family of sets. We omit their easy formulations.

Remark 8. The obtained generalizations of the Minkowski inequality allow to introduce some new paranormed function spaces (cf. [9]).

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