Nonlinear Analysis 75 (2012) 4995–5001

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

# Uniformly bounded composition operators in the Banach space of absolutely continuous functions

D. Głazowska [a,*], J. Matkowski [a], N. Merentes [b], J.L. Sánchez Hernández [b]

[a] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4a, PL-65-516 Zielona Góra, Poland

[b] Departamento de Matemática, Universidad Central de Venezuela, Caracas 1220A, Venezuela

ARTICLE INFO

*Article history:*
Received 2 February 2012
Accepted 8 April 2012
Communicated by Enzo Mitidieri

*MSC:*
primary 47H30

*Keywords:*
Composition operator
Nemytskij operator
Bounded operator
Uniformly bounded operator
Equidistantly uniformly bounded operator
Lipschitzian operator
Uniformly continuous operator
Banach space of absolutely continuous functions

ABSTRACT

Let $I, J \subset \mathbb{R}$ be intervals and let $h : I \times J \to \mathbb{R}$ be an arbitrary function. Our main result says that if the Nemytskij composition operator $H$ of the generator $h$ mapping the set $AC(I, J)$ into the Banach space $AC(I, \mathbb{R})$ is uniformly bounded (or equidistantly uniformly bounded), then

$$h(x, y) = a(x)y + b(x), \quad x \in I, \ y \in \mathbb{R},$$

for some functions $a, b \in AC(I, \mathbb{R})$.



## 0. Introduction

Let $I, J \subset \mathbb{R}$ be intervals. By $J^I$ denote the set of all functions $\varphi : I \to J$. For a given function $h : I \times J \to \mathbb{R}$, the mapping $H : J^I \to \mathbb{R}^I$ defined by

$$H(\varphi)(x) := h(x, \varphi(x)), \quad \varphi \in J^I,$$

is called a composition (superposition or Nemytskij) operator of a generator $h$.

The composition operators play important roles in the theory of differential equations, integral equations and functional equations. It is known that every locally defined operator mapping the set of continuous functions $C(I, J)$ into $C(I, \mathbb{R})$ must be a composition operator (cf. [1–6]). Moreover $H$ maps $C(I, J)$ into $C(I, \mathbb{R})$ if, and only if, its generator $h$ is continuous (Krasnoselskij). Considering this background, it is surprising that there are discontinuous functions $h : I \times \mathbb{R} \to \mathbb{R}$ generating the composition operators $H$ which map the space of continuously differentiable functions $C^1(I, \mathbb{R})$ into itself (cf. [7,1]). In [8] (1982) it was proved that if a superposition operator maps the Banach space $Lip(I, \mathbb{R})$ into itself and is globally Lipschitzian with respect to the $Lip$-norm then its generator must be of the form

$$h(x, y) = a(x)y + b(x), \quad x \in I, \ y \in \mathbb{R}.$$

Then this result was extended to some other function Banach spaces (cf. [7] for other references).

* Corresponding author.
*E-mail addresses:* D.Glazowska@wmie.uz.zgora.pl (D. Głazowska), J.Matkowski@wmie.uz.zgora.pl (J. Matkowski), nmerucv@gmail.com (N. Merentes), casanay085@hotmail.com (J.L. Sánchez Hernández).

0362-546X/$ – see front matter 
doi:10.1016/j.na.2012.04.013

In [9] it is shown that the analogous result holds true for the Banach space $AC(I,\mathbb{R})$ of absolutely continuous functions $\varphi : I \to \mathbb{R}$ and in [10] it was proved that the Lipschitz continuity of the operator can be replaced by its uniform continuity.

In the present paper we significantly improve these results. Denote by $AC(I,J)$ the set of all functions $\varphi \in AC(I,\mathbb{R})$ such that $\varphi(I) \subset J$. Applying the theory of Jensen to the functional equation, in Section 1 we prove the following auxiliary result (Theorem 1): *if the operator H mapping the set $AC(I,J)$ into the space $AC(I,\mathbb{R})$ satisfies the inequality*

$$\|H(\varphi) - H(\psi)\|_{AC} \le \gamma(\|\varphi - \psi\|_{AC}), \quad \varphi, \psi \in AC(I,J),$$

*for a function $\gamma : [0,\infty) \to [0,\infty)$, then there are $a, b \in AC(I,\mathbb{R})$ such that*

$$h(x,y) = a(x)y + b(x), \quad x \in I,\ y \in J. \tag{I}$$

Recently Matkowski [11] introduced the following notions. An operator $H$ mapping a metric space $\mathcal{Y}$ into a metric space $\mathcal{Z}$ is called:

(i) (cf. Definition 1) *uniformly bounded* if for any $t > 0$ there is a nonnegative real number $\gamma$ such that for any nonempty set $B \subset \mathcal{Y}$, if diam $B = t$, then diam $H(B) \le \gamma$;

(ii) (cf. Definition 2) *equidistantly uniformly bounded* if this condition holds true for all two-point sets $B$.

In [11] it is proved that any uniformly bounded or equidistantly uniformly bounded composition operator acting between general Lipschitz function normed spaces must be of form (I). Let us note that actually, every bounded operator is uniformly bounded.

In Section 2 we prove the main result of the present paper (Theorem 2) which reads as follows: *if the operator H mapping the set $AC(I,J)$ into the space $AC(I,\mathbb{R})$ is uniformly bounded (or equidistantly uniformly bounded), then (I) holds true for some $a, b \in AC(I,\mathbb{R})$.*

We show that the Lipschitzian composition operator $H : AC(I,J) \to AC(I,\mathbb{R})$ generated by the function $h(x,y) = y^2$ is locally Lipschitzian (using an example).

## 1. An auxiliary result and its consequences

Let $I \subset \mathbb{R}$ be a compact interval and let $x_0 \in I$. Then the set $AC(I,\mathbb{R})$ of all absolutely continuous functions $\varphi : I \to \mathbb{R}$ with the norm

$$\|\varphi\|_{AC} := |\varphi(x_0)| + \int_I |\varphi'(t)|dt$$

is a Banach space.

We begin with the following.

**Lemma 1.** *Let $I, J \subset \mathbb{R}$ be intervals and $h : I \times J \to \mathbb{R}$ be an arbitrary function. If the composition operator H of the generator h maps the set $AC(I,J)$ into the set $C(I,\mathbb{R})$ of continuous functions, then h is continuous in $I \times J$.*

**Proof.** Assume that $H(AC(I,J)) \subset C(I,\mathbb{R})$. The continuity of $h$ will be verified if we prove that for every sequence $((x_n,y_n))_{n\in\mathbb{N}}$ in $I \times J$ convergent to $(x,y) \in I \times J$, the sequence $(h(x_n,y_n))_{n\in\mathbb{N}}$ has a subsequence convergent to $h(x,y)$.

So, let $((x_n,y_n))_{n\in\mathbb{N}}$ be a sequence in $I \times J$ convergent to $(x,y) \in I \times J$.

*Case* 1. $x_n \neq x$ for every $n \in \mathbb{N}$.

It is well known and easy to see that if $(t_n)_{n\in\mathbb{N}}$ is a sequence of real numbers convergent to $t \in \mathbb{R}$, then $(t_n)_{n\in\mathbb{N}}$ has a monotonic subsequence. If, additionally, $t_n \neq t$ for infinitely many $n$, $(t_n)_{n\in\mathbb{N}}$ has a strictly monotonic subsequence. Thus, passing to a subsequence, we can, without loss of generality, assume that $(x_n)_{n\in\mathbb{N}}$ is strictly monotonic and $(y_n)_{n\in\mathbb{N}}$ is strictly monotonic. Let $I_0$ be a closed subinterval of $I$ with the ends $x_1$ and $x$. Define $\varphi_0 : I_0 \to J$ in the following way: $\varphi_0(x) = y$ and the graph of $\varphi_0$ restricted to the closed interval with the ends $x_n$ and $x_{n+1}$ is the line segment connecting the points $(x_n, y_n)$ and $(x_{n+1}, y_{n+1})$. Taking into account the facts that $(x_n)_{n\in\mathbb{N}}$ is strictly monotonic, $(y_n)_{n\in\mathbb{N}}$ is monotonic, $x_n \to x$ and $y_n \to y$, we see that $\varphi_0$ is well-defined, continuous and monotonic. Now we extend $\varphi_0 : I_0 \to J$ to a continuous function $\varphi : I \to J$ by setting $\varphi$ as an appropriate constant on each of the two intervals comprising $I \setminus I_0$. For example, $\varphi(t) = y$ for $t < x$ if $x < x_1$. Clearly, $\varphi$ is also monotonic. Furthermore, it is absolutely continuous:

$$\int_I |\varphi'|dt = \sum_{n=1}^{\infty} \int_{[x_{n+1},y_{n+1}]} |\varphi'|dt = \sum_{n=1}^{\infty} |y_{n+1} - y_n| = |y_1 - y|.$$

Hence $H(\varphi)(x) = h(x,\varphi(x))$ is continuous and therefore $H(\varphi)(x_n) \to H(\varphi)(x)$. On the other hand, $H(\varphi)(x_n) = h(x_n,\varphi(x_n)) = h(x_n,y_n)$ and $H(\varphi)(x) = h(x,\varphi(x)) = h(x,y)$. Hence $h(x_n,y_n) \to h(x,y)$, as required.

*Case* 2. The general case. If there are infinitely many $n$ for which $x_n \neq x$, we can, passing to a subsequence, move into the jurisdiction of Case 1. Otherwise, we can, without loss of generality, assume that $x_n = x$ for each $n \in \mathbb{N}$. Using the inclusion $H(AC(I,J)) \subset C(I,\mathbb{R})$ and applying $H$ to constant functions, one sees that $H$ is continuous with respect to the first variable. Thus, we can find a sequence $(u_n)$ in $I$ such that $u_n \neq x$ for every $n$, $u_n \to x$ and $(h(x,y_n) - h(u_n,y_n)) \to 0$. Applying Case 1 to the sequence $((u_n,y_n))$, we find that $(h(u_n,y_n))$ has a subsequence $(h(u_{n_k},y_{n_k}))$ convergent to $h(x,y)$. Since $(h(x,y_{n_k}) - h(u_{n_k},y_{n_k})) \to 0$, we have $h(x_{n_k},y_{n_k}) = h(x,y_{n_k}) \to h(x,y)$, as required. □

**Theorem 1.** *Let $I, J \subset \mathbb{R}$ be intervals and $h : I \times J \to \mathbb{R}$ be an arbitrary function. If there exists a function $\gamma : [0, \infty) \to [0, \infty)$ such that the composition operator $H$ of the generator $h$ maps the set $AC(I, J)$ into the Banach space $AC(I, \mathbb{R})$ and satisfies the inequality*

$$\|H(\varphi) - H(\psi)\|_{AC} \leq \gamma(\|\varphi - \psi\|_{AC}), \quad \varphi, \psi \in AC(I, J), \tag{1}$$

*then*

$$h(x, y) = a(x)y + b(x), \quad x \in I,\ y \in J,$$

*for some functions $a, b \in AC(I, \mathbb{R})$.*

**Proof.** In view of Lemma 1, the function $h$ is continuous in $I \times J$.

Without any loss of generality we can assume that $I = [0, 1]$ and that

$$\|\varphi\|_{AC} := |\varphi(0)| + \int_0^1 |\varphi'(t)|dt.$$

Take arbitrary $n \in \mathbb{N}$, $x \in I$, $y_1, y_2, \bar{y}_1, \bar{y}_2 \in J$, and a finite sequence $x_1, x_2, \ldots, x_{2n}$ such that

$$0 < x_1 < x_2 < \cdots < x_{2n} < 1.$$

Let $\varphi : I \to J$ be the polygonal function whose graph is uniquely determined by the vertices

$$(0, y_1), (x_1, y_1), (x_2, y_2), \ldots, (x_{2k-1}, y_1), (x_{2k}, y_2), \ldots, (x_{2n}, y_2), (1, y_2)$$

and, similarly, let $\psi : I \to J$ be the polygonal function whose graph is uniquely determined by the vertices

$$(0, \bar{y}_1), (x_1, \bar{y}_1), (x_2, \bar{y}_2), \ldots, (x_{2k-1}, \bar{y}_1), (x_{2k}, \bar{y}_2), \ldots, (x_{2n}, \bar{y}_2), (1, \bar{y}_2).$$

Clearly, $\varphi, \psi \in AC(I, J)$. Since $\varphi$ and $\psi$ are constant in the intervals $[0, x_1]$ and $[x_{2n}, 1]$, and affine in each of the intervals $[x_k, x_{k+1}]$, $k = 1, \ldots, 2n - 1$, by the definition of the norm $\|\cdot\|_{AC}$, we have

$$\begin{aligned}
\|\varphi - \psi\|_{AC} &= |\varphi(0) - \psi(0)| + \int_0^1 |\varphi'(t) - \psi'(t)|dt \\
&= |y_1 - \bar{y}_1| + \sum_{k=1}^{2n-1} \int_{x_k}^{x_{k+1}} |\varphi'(t) - \psi'(t)|dt \\
&= |y_1 - \bar{y}_1| + \sum_{k=1}^{2n-1} |y_1 - \bar{y}_1 - y_2 + \bar{y}_2| \\
&= |y_1 - \bar{y}_1| + (2n - 1)|y_1 - \bar{y}_1 - y_2 + \bar{y}_2|.
\end{aligned}$$

Moreover

$$\begin{aligned}
\|H(\varphi) - H(\psi)\|_{AC} &= |h(0, \varphi(0)) - h(0, \psi(0))| + \int_0^1 \left| \frac{d}{dt}[h(t, \varphi(t)) - h(t, \psi(t))] \right| dt \\
&\geq \int_0^1 \left| \frac{d}{dt}[h(t, \varphi(t)) - h(t, \psi(t))] \right| dt \\
&= \sum_{k=1}^{2n-1} \int_{x_k}^{x_{k+1}} \left| \frac{d}{dt}[h(t, \varphi(t)) - h(t, \psi(t))] \right| dt \\
&\geq \sum_{k=1}^{2n-1} \left| \int_{x_k}^{x_{k+1}} \frac{d}{dt}[h(t, \varphi(t)) - h(t, \psi(t))]dt \right| \\
&= \sum_{k=1}^{2n-1} |h(x_{k+1}, \varphi(x_{k+1})) - h(x_{k+1}, \psi(x_{k+1})) - h(x_k, \varphi(x_k)) + h(x_k, \psi(x_k))|.
\end{aligned}$$

Note that for each $k \in \{1, 2, \ldots, 2n\}$ either $\varphi(x_k) = y_1$ or $\varphi(x_k) = y_2$ and

$$\varphi(x_k) = y_1 \iff \varphi(x_{k+1}) = y_2, \quad k \in \{1, 2, \ldots, 2n - 1\}.$$

These equivalences are true if we replace $\varphi$ by $\psi$, $y_1$ by $\bar{y}_1$ and $y_2$ by $\bar{y}_2$.

Hence, applying inequality (1), we obtain

$$\sum_{k=1}^{2n-1} |h(x_{k+1}, y_1) - h(x_{k+1}, \bar{y}_1) - h(x_k, y_2) + h(x_k, \bar{y}_2)| \leq \gamma(|y_1 - \bar{y}_1| + (2n - 1)|y_1 - \bar{y}_1 - y_2 + \bar{y}_2|).$$

Letting $x_k$ tend to $x$ for all $k \in \{1, 2, \dots, 2n\}$ and making use of the continuity of $h$ in $I \times J$, we hence get

$$(2n-1)|h(x, y_1) - h(x, \bar{y}_1) - h(x, y_2) + h(x, \bar{y}_2)| \le \gamma(|y_1 - \bar{y}_1| + (2n-1)|y_1 - \bar{y}_1 - y_2 + \bar{y}_2|)$$

for all $n \in \mathbb{N}, x \in I, y_1, y_2, \bar{y}_1, \bar{y}_2 \in J$.

Taking arbitrary $u, v \in J$ and substituting here

$$y_1 = \bar{y}_2 := \frac{u+v}{2}, \qquad y_2 := u, \qquad \bar{y}_1 := v,$$

we obtain

$$(2n-1)\left|h\left(x, \frac{u+v}{2}\right) - h(x, u) - h(\bar{x}, v) + h\left(\bar{x}, \frac{u+v}{2}\right)\right| \le \gamma\left(\frac{|u-v|}{2}\right)$$

for all $x, \bar{x} \in I, u, v \in J$.

Letting here $\bar{x}$ tend to $x$ and making use of the continuity of $h$, we hence get

$$(2n-1)\left|2h\left(x, \frac{u+v}{2}\right) - h(x, u) - h(x, v)\right| \le \gamma\left(\frac{|u-v|}{2}\right)$$

for all $n \in \mathbb{N}, x \in I, u, v \in J$. Since $n \in \mathbb{N}$ is arbitrary, it follows that

$$2h\left(x, \frac{u+v}{2}\right) = h(x, v) + h(x, u), \quad x \in I, u, v \in J,$$

that is, for every $x \in I$, the function $h(x, \cdot)$ satisfies the Jensen functional equation. The continuity of $h$ implies that (cf. [12], p. 315, Theorems 1 and 2, p. 218) for every $x \in I$ there are $a(x), b(x) \in \mathbb{R}$ such that

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in J.$$

Since $h(\cdot, c) \in AC(I, \mathbb{R})$ for every $c \in J$, the functions $a$ and $b$ are absolutely continuous. □

For $\gamma(t) = Lt \, (t \ge 0)$ we get the result of [9].

**Remark 1.** Assuming additionally that the function $\gamma : [0, \infty) \to [0, \infty)$ is increasing, Theorem 1 remains valid if we replace the norm $\|\cdot\|_{AC}$ by an equivalent one. Taking a compact interval $I \subset \mathbb{R}$ and setting

$$\|\varphi\| := \sup_{x \in I} |\varphi(x)| + \int_I |\varphi'(t)| dt, \tag{2}$$

we have

$$\|\varphi\|_{AC} \le \|\varphi\| \le 2\|\varphi\|_{AC},$$

so the norms $\|\cdot\|_{AC}$ and $\|\cdot\|$ are equivalent. It easy to check the following: if $a, b \in AC(I, \mathbb{R})$ then the superposition operator $H$ of the generator $h(x, y) = a(x)y + b(x)$ $(x \in I, \ y \in \mathbb{R})$ maps the Banach space $AC(I, \mathbb{R})$ into itself, and

$$\|H(\varphi) - H(\psi)\|_{AC} \le \|a\| \|\varphi - \psi\|_{AC}, \quad \varphi, \psi \in AC(I, \mathbb{R}),$$

that is, $H$ is Lipschitzian.

**Corollary 1.** *Let $I, J \subset \mathbb{R}$ be intervals, $I$ compact, and $h : I \times J \to \mathbb{R}$. Suppose that the superposition operator $H$ of the generator $h$ maps the set $AC(I, J)$ into the Banach space $AC(I, \mathbb{R})$. Then $H$ is uniformly continuous if, and only if, there exist $a, b \in AC(I, \mathbb{R})$ such that*

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in J.$$

**Proof.** By Lemma 1, $h$ is continuous in $I \times J$. We may assume that $I = [0, 1]$. Suppose that $H$ is uniformly continuous. Then for every $\varepsilon > 0$ there is a finite $\delta = \delta(\varepsilon) > 0$ such that for all $\varphi, \psi \in AC(I, J)$,

$$\|\varphi - \psi\|_{AC} \le \delta \Longrightarrow \|H(\varphi) - H(\psi)\|_{AC} \le \varepsilon.$$

It follows that the function $\gamma : [0, \infty) \to [0, \infty)$,

$$\gamma(t) := \sup\{\|H(\varphi) - H(\psi)\|_{AC} : \|\varphi - \psi\|_{AC} \le t\}, \quad t \ge 0,$$

is correctly defined, $\gamma(0) = 0$, $\gamma$ is continuous at 0, increasing, and finite (as $\gamma(t) \le \delta(t) < \infty$), and we have

$$\|H(\varphi) - H(\psi)\|_{AC} \le \gamma(\|\varphi - \psi\|_{AC}), \quad \varphi, \psi \in AC(I, J).$$

Now the "only if" part of the corollary follows from Theorem 1. Since the "if" part is a consequence of Remark 1, the proof is complete. □

## 2. Bounded and uniformly bounded composition operators

Let us cite the following.

**Definition 1** (*Matkowski [11]*)**.** Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that a mapping $H : \mathcal{Y} \to \mathcal{Z}$ is uniformly bounded if for any $t > 0$ there is a nonnegative real number $\gamma(t)$ such that for any nonempty set $B \subset \mathcal{Y}$ we have

$$\operatorname{diam} B = t \Longrightarrow \operatorname{diam} H(B) \le \gamma(t).$$

**Remark 2.** Obviously, every uniformly continuous operator or Lipschitzian operator is uniformly bounded. Note that, under the assumptions of this definition, every bounded operator is uniformly bounded.

Applying Theorem 1 we prove:

**Theorem 2.** *Let $I, J \subset \mathbb{R}$ be intervals and $h : I \times J \to \mathbb{R}$ be an arbitrary function. If the composition operator $H$ of the generator $h$ mapping the set $AC(I, J)$ into the Banach space $AC(I, \mathbb{R})$ is uniformly bounded, then*

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in J,$$

*for some functions $a, b \in AC(I, \mathbb{R})$.*

**Proof.** Take arbitrary $\varphi, \psi \in AC(I, J)$ and put $t = \|\varphi - \psi\|_{AC}$. Since $\operatorname{diam}\{\varphi, \psi\} = t$, the uniform boundedness of $H$ implies that $\operatorname{diam} H(\{\varphi, \psi\}) \le \gamma(t)$, that is

$$\|H(\varphi) - H(\psi)\|_{AC} = \operatorname{diam} H(\{\varphi, \psi\}) \le \gamma(\|\varphi - \psi\|_{AC}),$$

and the result follows from Theorem 1. □

**Remark 3.** If the function $\gamma : [0, \infty) \to [0, \infty)$ in Definition 1 is continuous at 0 and $\gamma(0) = 0$, then, clearly, the uniform boundedness of the operator $H$ implies the uniform continuity of $H$.

It follows that Theorem 2 improves the result of [10] where $H$ is assumed to be uniformly continuous.
Consider the following.

**Definition 2** (*Matkowski [11]*)**.** Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that a mapping $H : \mathcal{Y} \to \mathcal{Z}$ is equidistantly uniformly bounded if for every $t > 0$ there is a nonnegative real number $\gamma(t)$ such that for all $u, v \in B \subset \mathcal{Y}$,

$$\operatorname{diam}\{u, v\} = t \Longrightarrow \operatorname{diam}\{H(u), H(v)\} \le \gamma(t).$$

Of course, the equidistant uniform boundedness is a weaker condition than the uniform boundedness. The following result follows from Theorem 1.

**Theorem 3.** *Let $I, J \subset \mathbb{R}$ be intervals and $h : I \times J \to \mathbb{R}$ be an arbitrary function. If the composition operator $H$ of the generator $h$ mapping the set $AC(I, J)$ into the Banach space $AC(I, \mathbb{R})$ is equidistantly uniformly bounded, then*

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in J,$$

*for some functions $a, b \in AC(I, \mathbb{R})$.*

**Example 1.** Take $I = J = [0, 1]$ and $h : I \times J \to \mathbb{R}$, $h(x, y) := y^2$. We shall show that for any $\varphi_0 \in AC(I, J)$ and for any $r > 0$ there is an $L > 0$ such that for all $\varphi, \psi \in AC(I, J)$,

$$\|\varphi - \varphi_0\| < r \quad \text{and} \quad \|\psi - \varphi_0\| < r \Longrightarrow \|H(\varphi) - H(\psi)\| \le L\|\varphi - \psi\|, \tag{3}$$

where the norm is defined by formula (2).

For the simplicity of calculation we assume that $\varphi_0 = 0$. Take arbitrary $r > 0$ and $\varphi, \psi \in AC(I, J)$ such that $\|\varphi\| < r$ and $\|\psi\| < r$. Since $|\varphi| = \varphi \le 1$ and $\int_0^1 |\psi'(x)|dx \le r$, we have

$$\begin{aligned}
\int_0^1 |(H(\varphi) - H(\psi))'|(x)dx &= \int_0^1 |([\varphi(x)]^2 - [\psi(x)]^2)'|(x)dx \\
&= 2\int_0^1 |\varphi(x)\varphi'(x) - \psi(x)\psi'(x)|dx \\
&\le 2\int_0^1 |\varphi(x)||\varphi' - \psi'|(x)dx + \int_0^1 |\psi'(x)||\varphi - \psi|(x)dx \\
&\le 2\int_0^1 |\varphi' - \psi'|(x)dx + 2\sup_{x\in[0,1]} |\varphi(x) - \psi(x)| \int_0^1 |\psi'(x)|dx \\
&\le 2\int_0^1 |(\varphi - \psi)'|(x)dx + 2r\sup_{x\in[0,1]} |\varphi(x) - \psi(x)|.
\end{aligned}$$

Moreover,

$$\begin{aligned}|H(\varphi)(x) - H(\psi)(x)| &= |[\varphi(x)]^2 - [\psi(x)]^2| = |\varphi(x) + \psi(x)||\varphi(x) - \psi(x)| \\ &\leq 2|\varphi(x) - \psi(x)|,\end{aligned}$$

whence

$$\sup_{x\in[0,1]} |H(\varphi)(x) - H(\psi)(x)| \leq 2 \sup_{x\in[0,1]} |\varphi(x) - \psi(x)|.$$

Hence, setting $L = \max(2(r + 1), 2)$, we get

$$\begin{aligned}\|H(\varphi) - H(\psi)\| &= \sup_{x\in[0,1]} |H(\varphi)(x) - H(\psi)(x)| + \int_0^1 |(H(\varphi) - H(\psi))'|(x)dx \\ &\leq 2(r + 1) \sup_{x\in[0,1]} |\varphi(x) - \psi(x)| + 2\int_0^1 |(\varphi - \psi)'|(x)dx \\ &\leq L\|\varphi - \psi\|,\end{aligned}$$

as required.

Since the norms $\|\cdot\|$ and $\|\cdot\|_{AC}$ are equivalent, the norm $\|\cdot\|_{AC}$ also satisfies condition (3).

Thus the composition operator $H$ of the generator $h$ is a locally Lipschitzian map from $AC(I, J)$ to $AC(I, \mathbb{R})$.

**Remark 4.** Let us mention that any composition operator $H : AC(I, J) \to AC(I, \mathbb{R})$ is locally defined (cf. [1–6]), which means that for all $\varphi, \psi \in AC(I, J)$ and every open interval $I_0 \subset I$, the equality $\varphi|_{I_0} = \psi|_{I_0}$ implies that $K(\varphi)|_{I_0} = K(\psi)|_{I_0}$.

## 3. Equivalent reformulations of inequality (1)

In this section we show that inequality (1), the basic assumption of Theorem 1, is rather a weak condition. For simplicity of consideration, we assume that $J = \mathbb{R}$.

We begin with the following.

**Remark 5.** Let $(V, \|\cdot\|)$ be a real normed space and $r$ be a positive real number. If $u, v \in V$ and $\|u - v\| \leq 2r$, then there is a $w \in V$ such that $\|u - w\| = \|v - w\| = r$.

It follows from the fact that two spheres centered at $u$ and $v$ of radius $r$ have nonempty intersection.

**Lemma 2.** *Let $H$ be a mapping of a real normed space $(V, \|\cdot\|)$ into itself. Assume that $M, r > 0$ and $\|H(u) - H(v)\| \leq M$ whenever $\|u - v\| = r$. Then $\|H(u) - H(v)\| \leq 2M$ whenever $\|u - v\| \leq 2r$.*

**Proof.** Let $\|u - v\| \leq 2r$. By Remark 5 there is a $w \in V$ such that $\|u - w\| = \|v - w\| = r$. Hence

$$\|H(u) - H(v)\| \leq \|H(u) - H(w)\| + \|H(w) - H(v)\| \leq 2M. \quad \square$$

Lemma 2 easily implies the following.

**Remark 6.** Let $I \subset \mathbb{R}$ be an interval and $h : I \times \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Assume that the composition operator $H$ of the generator $h$ maps the space $AC(I, \mathbb{R})$ into itself. Then the following conditions are equivalent:

(i) there exists a function $\gamma : [0, \infty) \to [0, \infty)$ such that

$$\|H(\varphi) - H(\psi)\|_{AC} \leq \gamma(\|\varphi - \psi\|_{AC}), \quad \varphi, \psi \in AC(I, \mathbb{R});$$

(ii) there exist $M, r > 0$ such that $\|H(\varphi) - H(\psi)\|_{AC} \leq M$ for all $\varphi, \psi \in AC(I, \mathbb{R})$ satisfying $\|\varphi - \psi\|_{AC} = r$;

(iii) there exist $M, r > 0$ such that $\|H(\varphi) - H(\psi)\|_{AC} \leq M$ for all $\varphi, \psi \in AC(I, \mathbb{R})$ satisfying $\|\varphi - \psi\|_{AC} \leq r$.

Note that condition (ii) is apparently weaker than (i) (it only requires $\gamma$ to be defined at one point $r$), while (iii) is apparently stronger than (i).

This discussion allows us to formulate the following.

**Corollary 2.** *Let $I \subset \mathbb{R}$ be an interval and $h : I \times \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Assume that the composition operator $H$ of the generator $h$ maps the space $AC(I, \mathbb{R})$ into itself. If there exist $M, r > 0$ such that $\|H(\varphi) - H(\psi)\|_{AC} \leq M$ for all $\varphi, \psi \in AC(I, \mathbb{R})$ satisfying $\|\varphi - \psi\|_{AC} = r$ then*

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in J,$$

*for some functions $a, b \in AC(I, \mathbb{R})$.*

## Acknowledgments

The authors wish to thank the reviewers for carefully reading the manuscript and indicating several improvements. We are especially indebted to them for Lemma 1, which allowed us to remove a regularity condition assumed in the original manuscript.

## References

[1] K. Lichawski, J. Matkowski, J. Miś, Locally defined operators in the space of differentiable functions, Bull. Polish Acad. Sci. Math. 37 (1989) 315–325.
[2] J. Matkowski, M. Wróbel, Locally defined operators in the space of Whitney differentiable functions, Nonlinear Anal. 68 (2008) 2933–2942.
[3] J. Matkowski, M. Wróbel, Representation theorem for locally defined operators in the space of Whitney differentiable functions, Manuscripta Math. 129 (2009) 437–448.
[4] M. Wróbel, Locally defined operators in Hölder's spaces, Nonlinear Anal. 74 (2011) 317–323.
[5] M. Wróbel, Representation theorem for local operators in the space of continuous and monotone functions, J. Math. Anal. Appl. 372 (2010) 45–54.
[6] M. Wróbel, Locally defined operators and a partial solution of a conjecture, Nonlinear Anal. 72 (2010) 495–506.
[7] J. Appell, P.P. Zabrejko, Nonlinear Superposition Operators, Cambridge University Press, Cambridge, 1990.
[8] J. Matkowski, Functional equations and Nemytskij operators, Funkc. Ekvacioj Ser. Int. 25 (1982) 127–132.
[9] J. Matkowski, Lipschitzian composition operators in some function spaces, Nonlinear Anal. 30 (1997) 719–726.
[10] J. Matkowski, Uniformly continuous superposition operators in the spaces of differentiable functions and absolutely continuous functions, Internat. Ser. Numer. Math. 157 (2008) 155–166.
[11] J. Matkowski, Uniformly bounded composition operators between general Lipschitz function normed spaces, Topol. Methods Nonlinear Anal. 38 (2011) 395–405.
[12] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, Polish Scientific Editors and Silesian University, Warszawa, Kraków, Katowice, 1985.