Chapter 37
On Means Which are Quasi-Arithmetic and of the Beckenbach–Gini Type

Janusz Matkowski

Dedicated to the memory of S.M. Ulam
on the 100th anniversary of his birth

Abstract The class of quasi-arithmetic means and the class of Beckenbach–Gini means are essentially different. The problem of characterization of the means which belong to both classes leads to a composite functional equation for two unknown functions. We solve this functional equation assuming that a generator of quasi-arithmetic mean is once continuously differentiable.

Keywords Quasi-arithmetic mean • Beckenbach–Gini mean • Functional equation

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37.1 Introduction

Let $I \subset \mathbb{R}$ be an interval. By a *mean* we mean any function $M : I^2 \to I$ such that

$$\min(x,y) \le M(x,y) \le \max(x,y), \qquad x,y \in I.$$

If for all $x,y \in I, x \neq y$, these inequalities are sharp, M is called a *strict mean*. If $M : I^2 \to I$ is a mean then, of course, M is *reflexive*, i.e.,

$$M(x,x) = x, \qquad x \in I.$$

J. Matkowski (✉)
Faculty of Mathematics Computer Science and Econometrics, University of Zielona Góra, Podgórna 50, PL-65-246 Zielona Góra, Poland

Institute of Mathematics, Silesian University, Bankowa 14, PL-42-007 Katowice, Poland
e-mail: J.Matkowski@wmie.uz.zgora.pl

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Every reflexive function $M : I^2 \to I$ which is increasing with respect to each variable is a mean. In this paper we call such functions the *increasing means*.

If $\varphi : I \to \mathbb{R}$ is continuous and strictly monotonic and $f : I \to (0,\infty)$ an arbitrary function then $M : I^2 \to I$ given by

$$M(x,y) := \varphi^{-1}\left(\frac{\varphi(x)f(x)+\varphi(y)f(y)}{f(x)+f(y)}\right), \qquad x,y \in I,$$

is a mean and it is called *weighted quasi-arithmetic.*

Losonczi [5] considered the problem of equality of two quasi-arithmetic weighted means. Assuming among others that some of the involved functions are sixth times differentiable, reduces the problem to a sixth order differential equation. Next, employing the software package Maple V, he got 32 families of solutions. (For the historical background of the problem cf. [5].)

In the present paper, we consider a special case of the problem considered by Losonczi. Namely, we examine when the classical *quasi-arithmetic means* and *Beckenbach–Gini means* (cf. Beckenbach [2], Gini [4]; also: Bullen, Mitrinović, Vasić [3, Chap. III, p. 189]) coincide. However, we assume that functions involved are only continuously differentiable, and we apply elementary methods.

Recall that a mean $M = M^{[\varphi]} : I^2 \to I$ is quasi-arithmetic if there is a continuous and strictly monotonic function $\varphi : I \to \mathbb{R}$, called a *generator* of M, such that

$$M^{[\varphi]}(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \qquad x,y \in I.$$

Of course, every quasi-arithmetic mean is increasing and continuous. A mean $M = M_f$ is called a Beckenbach–Gini mean if there is a function $f : I \to (0,\infty)$, a generator of the mean, such that

$$M_f(x,y) = \frac{xf(x)+yf(y)}{f(x)+f(y)}, \qquad x,y \in I,$$

(note that one can assume that f is of a constant sign). In general, a Beckenbach–Gini mean need be increasing and continuous. Clearly, both quasi-arithmetic and Beckenbach–Gini means are strict.

In Sect. 2 of the present paper, we examine when $M^{[\varphi]} = M_f$, i.e. when the Beckenbach–Gini and quasi-arithmetic means coincide. More exactly, we consider the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) = \frac{xf(x)+yf(y)}{f(x)+f(y)}, \qquad x,y \in I,$$

when $\varphi : I \to \mathbb{R}$ and $f : I \to (0,\infty)$ are the unknown functions. Under the assumption of the continuous differentiability of the unknown functions we show (Theorem 37.1) that the functions φ and f satisfy this equation if, and only if,

$$\varphi'(x) = cf(x)^2 , \qquad x \in I ,$$

and

$$f(x) = \frac{1}{\sqrt{|Ax^2+Bx+C|}} , \qquad x \in I ,$$

for some $A,B,C,c \in \mathbb{R}$, $C \neq 0 \neq c$. Applying this result we determine all families of pairs of functions (φ, f) satisfying the considered functional equation (Corollary 37.1) as well as the relevant means. In some cases, the generator φ can be the log composed with a homographic function or arctan composed with an affine function.

The case $I = \mathbb{R}$ is considered. In this context, the generalized geometric and harmonic means appear. Note that these means can be treated as the Beckenbach–Gini ones but they are not quasi-arithmetic. The problem of extension of quasi-arithmetic means is considered. We end this section with some remarks on the three parameter family of Beckenbach–Gini means $\{M_{A,B,C} : A,B,C \in \mathbb{R}, A^2+B^2+C^2 > 0\}$, with $M_{A,B,C} := M_f$, where f is given by

$$f(x) = \frac{1}{\sqrt{|Ax^2+Bx+C|}} ,$$

indicating some interesting relations among these means and the generalized harmonic and geometric means.

In Sect. 3, we consider a higher dimensional version of equation $M^{[\varphi]} = M_f$. More exactly, we examine the functional equation

$$\varphi^{-1}\left(\frac{\sum_{j=1}^{k}\varphi(x_k)}{k}\right) = \frac{\sum_{j=1}^{k} x_k f(x_k)}{\sum_{j=1}^{k} f(x_k)} , \qquad x_1,\ldots,x_k \in I ,$$

where $k \geq 3$ is a fixed positive integer. We show that a strictly monotonic continuously differentiable function $\varphi : I \to \mathbb{R}$ and a continuously differentiable function $f : I \to (0,\infty)$ satisfy this equation if, and only if, either

$$\varphi(x) = ax+b , \qquad f(x) = c , \qquad x \in I ,$$

or

$$\varphi(x) = \frac{a}{x+d} + b , \qquad f(x) = \frac{c}{x+d} , \qquad x \in I ,$$

for some $a,b,c,d \in \mathbb{R}$, $a \neq 0 \neq c$ (and the joint forms of both means $M^{[\varphi]}$ and M_f are given).

Let us mention that if one of the means $M^{[\varphi]}$ or M_f is positively homogeneous then the equation $M^{[\varphi]} = M_f$ is easy to solve (cf. [6, Theorem 6 and Corollary 3]).

37.2 The Functional Equation $M^{[\varphi]} = M_f$

Some properties of Beckenbach–Gini and quasi-arithmetic means can be found in [3, Chaps. III and IV]. We begin with recalling the following easy to verify

Remark 37.1. Let $I \subset \mathbb{R}$ be an interval and $f,g : I \to (0,\infty)$.Then $M_f = M_g$ if, and only if, $g = cf$ for some $c > 0$.

Remark 37.2. Let $I \subset \mathbb{R}$ be an interval and $\varphi, \psi : I \to \mathbb{R}$ continuous and strictly monotonic. Then $M^{[\varphi]} = M^{[\psi]}$ if, and only if, $\psi = af + b$ for some $a,b \in \mathbb{R}$, $a \neq 0$.

Let us also note that every mean has the following *mean property:* if $M : I^2 \to I$ is a mean and $J \subset I$ is an interval, then M restricted to the set J^2 is a mean on J^2 and, consequently, $M(J^2) = J$.

The main result reads as follows

Theorem 37.1. *Let $I \subset \mathbb{R}$ be an open interval. A strictly monotonic continuously differentiable function $\varphi : I \to \mathbb{R}$ and a function $f : I \to (0,\infty)$ satisfy the functional equation $M^{[\varphi]} = M_f$ in I, i.e.,*

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) = \frac{xf(x)+yf(y)}{f(x)+f(y)}\,, \qquad x,y \in I\,, \tag{37.1}$$

if, and only if,

$$\varphi'(x) = cf(x)^2\,, \qquad x \in I\,,$$

and

$$f(x) = \frac{1}{\sqrt{|Ax^2+Bx+C|}}\,, \qquad x \in I\,,$$

for some $A,B,C,c \in \mathbb{R}$, $C \neq 0 \neq c$.

Proof. Suppose that a strictly monotonic and continuously differentiable function $\varphi : I \to \mathbb{R}$ and a function $f : I \to (0,\infty)$ satisfy equation 37.1.

Assume first that

$$\varphi'(x) \neq 0\,, \qquad x \in I\,.$$

Then (37.1) easily implies that f is also continuously differentiable in I. Write (37.1) in the form

$$\varphi(x)+\varphi(y) = 2\varphi\left(\frac{xf(x)+yf(y)}{f(x)+f(y)}\right)\,, \qquad x,y \in I\,. \tag{37.2}$$

Differentiating both sides of (37.2) with respect to x and y, respectively, gives

$$\varphi'(x) = 2\varphi'\left(\frac{xf(x)+yf(y)}{f(x)+f(y)}\right)\frac{f(x)^2+f(x)f(y)+xf'(x)f(y)-yf'(x)f(y)}{(f(x)+f(y))^2} \tag{37.3}$$

and

$$\varphi'(y) = 2\varphi'\left(\frac{xf(x)+yf(y)}{f(x)+f(y)}\right)\frac{f(y)^2+f(y)f(x)+yf'(y)f(x)-xf'(y)f(x)}{(f(x)+f(y))^2} \tag{37.4}$$

for all $x, y \in I$. Subtracting these equations by sides and then dividing by $x-y$, for all $x, y \in I$, $x \neq y$, we obtain

$$\frac{\varphi'(x)-\varphi'(y)}{x-y} = 2\varphi'\left(\frac{xf(x)+yf(y)}{f(x)+f(y)}\right)\frac{\dfrac{f(x)^2-f(y)^2}{x-y}+f'(x)f(y)+f(x)f(y)'}{(f(x)+f(y))^2}\,.$$

Letting here $y \to x$ we see that the limit of the right hand side exists and, consequently,

$$\varphi''(x) = 2\varphi'(x)\frac{f'(x)}{f(x)}\,, \qquad x \in I\,.$$

It follows that the function φ is twice continuously differentiable in I. Writing this relation in the form

$$\frac{\varphi''(x)}{\varphi'(x)} = 2\frac{f'(x)}{f(x)}\,, \qquad x \in I\,,$$

we infer that there exists a number $c \in \mathbb{R}$ such that

$$\varphi'(x) = cf(x)^2\,, \qquad x \in I\,. \tag{37.5}$$

Note that $c \neq 0$, as, by assumptions, f has positive values and φ is strictly monotonic. It follows that

$$\varphi'(x) \neq 0\,, \qquad x \in I\,,$$

and, by (37.1), the function f is also twice continuously differentiable in I.

Now from (37.3) and (37.4) we have

$$\frac{\varphi'(x)}{\varphi'(y)} = \frac{f(x)^2+f(x)f(y)+xf'(x)f(y)-yf'(x)f(y)}{f(y)^2+f(y)f(x)+yf'(y)f(x)-xf'(y)f(x)}\,, \qquad x, y \in I\,.$$

Hence, making use of (37.5), we obtain

$$\frac{f(x)^2}{f(y)^2} = \frac{f(x)^2+f(x)f(y)+xf'(x)f(y)-yf'(x)f(y)}{f(y)^2+f(y)f(x)+yf'(y)f(x)-xf'(y)f(x)}\,, \qquad x, y \in I\,,$$

which is equivalent to the relation

$$f(x)f(y)\left[f(x)^2 - f(y)^2\right] = (x-y)\left[f'(y)f(x)^3 - f'(x)f(y)^3\right]\,, \qquad x, y \in I\,. \tag{37.6}$$

Without any loss of generality, we can assume that

$$0 \in I .$$

By Remark 1 we can also assume that

$$f(0) = 1 .$$

Now, setting $y = 0$ in (37.5) gives

$$f'(x) = \frac{f(x)^3 - f(x)}{x} + af(x)^3 , \qquad x \in I, x \neq 0 . \tag{37.7}$$

Differentiating both sides of (37.6) with respect to y we obtain, for all $x, y \in I$,

$$\begin{aligned} &f(x)f'(y)\left[f(x)^2 - f(y)^2\right] - 2f(x)f(y)^2 f'(y) \\ &\quad = -\left[f'(y)f(x)^3 + f'(x)f(y)^3\right] + (x-y)\left[f''(y)f(x)^3 + 3f'(x)f(y)^2 f'(y)\right] . \end{aligned}$$

Putting here $y := 0$ gives

$$(3ax - 1)f'(x) = (2a - bx)f(x)^3 - 3af(x) , \qquad x \in I ,$$

where

$$a := f'(0), \qquad b := f''(0) .$$

Making use of (37.7), we hence get, for all $x \in I$, $x \neq 0$,

$$(3ax - 1)\left[\frac{f(x)^3 - f(x)}{x} + af(x)^3\right] = (2a - bx)f(x)^3 - 3af(x) ,$$

which, as f is positive and continuous, implies that

$$(3ax - 1)\left[f(x)^2 - 1 + axf(x)^2\right] = (2a - bx)xf(x)^2 - 3ax , \qquad x \in I ,$$

and, consequently,

$$f(x)^2 = \frac{1}{(3a^2 - b)x^2 - 2ax + 1} , \qquad x \in I .$$

Since, by the assumption, f is positive, we get

$$f(x) = \frac{1}{\sqrt{|(3a^2 - b)x^2 - 2ax + 1|}} , \qquad x \in I .$$

Thus, we have shown that f must be of the form

$$f(x) = \frac{1}{\sqrt{|Ax^2 + Bx + C|}}\,, \qquad x \in I\,, \tag{37.8}$$

for some $A, B, C \in \mathbb{R}$, such that $C \neq 0$.

To finish this part of the proof put

$$Z := \{x \in I : \varphi'(x) = 0\}\,.$$

The continuity of φ' implies that Z is closed. By the strict monotonicity of φ, the interior of Z is empty. Moreover, there exists at most countable set S such that

$$I \setminus Z = \bigcup_{s \in S} I_s\,,$$

where $\{I_s : s \in S\}$ is a family of open (in I) and disjoint intervals. If Z were not empty then there would exist an $s \in S$ such that

$$I_s = (x_0, x_1)\,, x_0 < x_1\,, \text{ with } x_0 \in Z \text{ or } x_1 \in Z\,.$$

Assume, for instance, that $x_0 \in Z$. Since $\varphi' \neq 0$ in I_s, according to what we have already shown,

$$\varphi'(x) = c f(x)^2\,, \qquad x \in (x_0, x_1)\,.$$

Equation (37.1) and the assumptions of φ imply that f is continuous in I. In particular, f is continuous at x_0. Letting $x \to x_0$ we hence get

$$0 = \varphi'(x_0) = c f(x_0)^2\,,$$

and, consequently, $f(x_0) = 0$. This contradiction shows that the set Z is empty, and completes the proof of the "only if" part of the theorem.

Making use formula (37.5), and considering several obvious cases (cf. Corollary 37.1 below), it is easy to verify that the converse implication holds true. □

Remark 37.3. Note that a function $f : I \to (0, \infty)$ satisfies (37.6) if, and only if, f is of the form (37.8).

Applying Theorem 37.1 we obtain

Corollary 37.1. *Let $I \subset \mathbb{R}$ be an interval. A strictly monotonic continuously differentiable function $\varphi : I \to \mathbb{R}$ and a continuously differentiable function $f : I \to (0, \infty)$ satisfy the functional equation*

$$M^{[\varphi]}(x, y) = M_f(x, y)\,, \qquad x, y \in I\,,$$

if, and only if, one of the following cases occurs:

1. *there are* $p,q,r,\ a,b\in\mathbb{R}$, $p<q$, $r>0$, $a\neq 0$, *such that* $I\subset(p,q)$,

$$f(x)=\frac{r}{\sqrt{(x-p)(q-x)}}\,,\qquad \varphi(x)=a\log\frac{x-p}{q-x}+b\,,\qquad x\in I\,;$$

moreover

$$M^{[\varphi]}(x,y)=\frac{x\sqrt{(y-p)(q-y)}+y\sqrt{(x-p)(q-x)}}{\sqrt{(y-p)(q-y)}+\sqrt{(x-p)(q-x)}}=M_f(x,y)\,,\qquad x,y\in I\,;$$

2. *there are* $p,q,r,\ a,b\in\mathbb{R}$, $p<q$, $r>0$, $a\neq 0$, *such that either* $I\subset(-\infty,p)$ *or* $I\subset(q,\infty)$, *and*

$$f(x)=\frac{r}{\sqrt{(x-p)(x-q)}}\,,\qquad \varphi(x)=a\log\frac{x-p}{q-x}+b\,,\qquad x\in I\,;$$

moreover

$$M^{[\varphi]}(x,y)=\frac{x\sqrt{(y-p)(y-q)}+y\sqrt{(x-p)(x-q)}}{\sqrt{(y-p)(y-q)}+\sqrt{(x-p)(x-q)}}=M_f(x,y)\,,\qquad x,y\in I\,;$$

3. *there are* $p,r,\ a,b\in\mathbb{R}$, $r>0$, $a\neq 0$, *such that either* $I\subset(-\infty,p)$ *or* $I\subset(p,\infty)$, *and*

$$f(x)=\frac{r}{\sqrt{(x-p)^2}}=\frac{r}{|x-p|}\,,\qquad \varphi(x)=\frac{a}{x-p}+b\,,\qquad x\in I\,;$$

moreover

$$M^{[\varphi]}(x,y)=p+\frac{2(x-p)(y-p)}{(x-p)+(y-p)}=M_f(x,y)\,,\qquad x,y\in I\,;$$

4. *there are* $p,q,r,\ a,b\in\mathbb{R}$, $p^2-4q<0$, $r>0$, $a\neq 0$, *such that*

$$f(x)=\frac{r}{\sqrt{x^2+px+q)}}\,,\qquad \varphi(x)=a\arctan\frac{2x+q}{\sqrt{4q-p^2}}+b\,,\qquad x\in I\,;$$

moreover

$$M^{[\varphi]}(x,y)=\frac{x\sqrt{y^2+py+q}+y\sqrt{x^2+px+q}}{\sqrt{y^2+py+q}+\sqrt{x^2+px+q}}=M_f(x,y)\,,\qquad x,y\in I\,;$$

5. *there are* $p,r,\ a,b\in\mathbb{R}$, $r>0$, $a\neq 0$, *such that either* $I\subset(-\infty,p)$,

$$f(x)=\frac{r}{\sqrt{p-x}}\,,\qquad \varphi(x)=a\log(p-x)+b\,,\qquad x\in I\,,$$

and moreover

$$M^{[\varphi]}(x,y) = -\sqrt{(x-p)(y-p)} + p = \frac{x\sqrt{p-y}+y\sqrt{p-x}}{\sqrt{p-y}+\sqrt{p-x}} = M_f(x,y)\,, \quad x,y \in I\,;$$

or $I \subset (p,\infty)$,

$$f(x) = \frac{r}{\sqrt{x-p}}\,, \qquad \varphi(x) = a\log(x-p)+b\,, \qquad x \in I\,,$$

and moreover

$$M^{[\varphi]}(x,y) = \sqrt{(x-p)(y-p)} + p = \frac{x\sqrt{y-p}+y\sqrt{x-p}}{\sqrt{y-p}+\sqrt{x-p}} = M_f(x,y),\ \ x,y \in I\,;$$

6. *there are* $r,\ a,b \in \mathbb{R}$, $r > 0$, $a \neq 0$, *such* $f(x) = r$ *and* $\varphi(x) = ax+b$ *for* $x \in I$*; moreover,*

$$M^{[\varphi]}(x,y) = \frac{x+y}{2} = M_f(x,y)\,, \qquad x,y \in I\,.$$

Remark 37.4. Note the following facts.

1. *In the cases 4 and 6 of Corollary 37.1, taking* $I = \mathbb{R}$ *gives the suitable Beckenbach–Gini and quasi-arithmetic means which are globally defined on* $\mathbb{R}^2$.
2. *Let*

$$g(x) := \frac{1}{\sqrt{|(x-p)(x-q)|}}\,, \qquad x \in \mathbb{R}\backslash\{p,q\}\,,$$

where $p \neq q$, *and define* $F_{p,q} : (\mathbb{R}^2\backslash A) \to \mathbb{R}$ *where* $A := \{p,q\}\times\mathbb{R}\cup\mathbb{R}\times\{p,q\}$ *by*

$$F_{p,q}(x,y) := \frac{x\sqrt{|(y-p)(y-q)|}+y\sqrt{|(x-p)(x-q)|}}{\sqrt{|(y-p)(y-q)|}+\sqrt{|(x-p)(x-q)|}}\,.$$

Since, for every $y \in \mathbb{R}\backslash\{p,q\}$,

$$\lim_{x\to p} F_{p.q}(x,y) = \lim_{x\to p} F(x,y) = p\,, \quad \lim_{x\to q} F_{p,q}(x,y) = \lim_{x\to q} F(x,y) = q\,, \tag{37.9}$$

the function $F_{p,q}$ *is not extendable to a continuous mean on* $\mathbb{R}^2$*. Note that the Beckenbach–Gini means mentioned in the cases 1 and 2 of Corollary 37.1 coincide with* $F_{p,q}$ *on some proper subsets of* $\mathbb{R}^2$.
3. *The mean considered in case 3 (where* $p = q$*), which is a translated harmonic one, is extendable to a generalized harmonic mean* $H_p : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $H_p(p,p) := p$ and

$$H_p(x,y) := p + \frac{(x-p)\,|y-p| + (y-p)\,|x-p|}{|y-p|+|x-p|}\,, \quad (x,y) \neq (p,p)\,.$$

This mean is continuous on $\mathbb{R}^2$ *and* $H_p(x,y) = p$ *for all* x, y *with* $xy \le 0$. *Since*

$$H_p(x,y) := \frac{x\,|y-p| + y\,|x-p|}{|y-p| + |x-p|}, \quad (x,y) \ne (p,p)\,,$$

H_p *can be treated as a Beckenbach–Gini mean of the generator* $f(x) = 1/\,|x-p|$.

4. *Similarly, the mean considered in case 5 (where* $p = q$*), a translated geometric one, is extendable to a generalized geometric mean* $G_p : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $H_p(p,p) := p$ and

$$H_p(x,y) := p + \frac{(x-p)\sqrt{|y-p|} + (y-p)\sqrt{|x-p|}}{\sqrt{|y-p|} + \sqrt{|x-p|}}, \quad (x,y) \ne (p,p)\,.$$

This mean is continuous on $\mathbb{R}^2$ *and* $G_p(x,y) = 0$ *for all* x, y *with* $xy \le 0$. *Since*

$$H_p(x,y) := \frac{x\sqrt{|y-p|} + y\sqrt{|x-p|}}{\sqrt{|y-p|} + \sqrt{|x-p|}}, \qquad (x,y) \ne (p,p)\,,$$

so H_p *can be treated as a Beckenbach–Gini mean of the generator* $f(x) = 1/\sqrt{|x-p|}$.

Remark 37.5. For every $p \in \mathbb{R}$, the generalized harmonic and geometric means, H_p and G_p, are unique increasing means which coincide with the translated harmonic and geometric means, respectively.

Proof. Since for all x, y such that $(x-p) + (y-p) > 0$,

$$H_p(x,y) = p + \frac{2(x-p)(y-p)}{(x-p)+(x-p)}, \qquad G_p(x,y) := p + \sqrt{(x-p)(y-p)}\,,$$

and for all x, y such that $(x-p) + (y-p) < 0$,

$$H_p(x,y) = p + \frac{2(p-x)(p-y)}{(p-x)+(p-y)}, \qquad G_p(x,y) := p + \sqrt{(p-x)(p-y)}\,,$$

the means H_p and G_p on the sets $(p,\infty)^2$ and $(-\infty,p)^2$ coincide, respectively, with the translated harmonic and geometric means.

Now suppose that $M : \mathbb{R}^2 \to \mathbb{R}$ is an increasing mean which coincides with the harmonic translated means on $(p,\infty)^2$ and $(-\infty,p)^2$. Then for all x, y such that $x \le p \le y$ we have

$$p = H_p(x,p) = M(x,p) \le M(x,y) \le M(p,y) = H_p(p,y) = p\,,$$

which proves that $M(x,y) = p$ for all $x \le p \le y$. □

As we have observed, the generalized harmonic and geometric means are of the Beckenbach–Gini type. However, we have the following.

Remark 37.6. The generalized harmonic and geometric means are not quasi-arithmetic ones.

This is a consequence of the subsequent

Proposition 37.1. *Let I ,$J \subset \mathbb{R}$ be intervals such that $I = (a,b)$, $J = (b,c)$ for some a,b,c, $-\infty \le a < b < c \le \infty$. Suppose that $M^{[\psi]} : I^2 \to I$ and $M^{[\gamma]} : J^2 \to J$ are quasi-arithmetic means of the suitable generators $\psi : I \to \mathbb{R}$ and $\gamma : J \to \mathbb{R}$. The following two conditions are equivalent:*

1. *there exists a quasi-arithmetic mean $M^{[\varphi]} : (a,c)^2 \to (a,c)$ such that*

$$M^{[\varphi]}(x,y) = M^{[\psi]}(x,y)\,, \quad x,y \in I\,; \qquad M^{[\varphi]}(x,y) = M^{[\gamma]}(x,y)\,, \quad x,y \in J\,;$$

2. *the limits*

$$\psi(b-) := \lim_{x\to b-} \psi(x)\,, \qquad \gamma(b+) := \lim_{x\to b+} \gamma(x)$$

exist and are finite.

Proof. It is an immediate consequence of the definition of a quasi-arithmetic mean and Remark 37.2. □

Take arbitrarily fixed $A,B,C \in \mathbb{R}$ such that $A^2+B^2+C^2 > 0$, and put $M_{A,B,C} := M_f$, where f is given by $f(x) = 1/\sqrt{|Ax^2+Bx+C|}$. The next remark shows an interesting relations of the family of Beckenbach–Gini means $M_{A,B,C}$ and the generalized harmonic and geometric means for $p = 0$.

Remark 37.7. The three-parameter family of means

$$\{M_{A,B,C} : A,B,C \in \mathbb{R},\ A^2+B^2+C^2 > 0\}$$

has the following properties: for all $x,y \in \mathbb{R}$,

$$\lim_{A\to\infty} M_{A,B,C}(x,y) = \lim_{A\to-\infty} M_{A,B,C}(x,y) = \frac{x\,|y| + y\,|x|}{|y| + |x|} = H_0(x,y)\,,$$

$$\lim_{B\to\infty} M_{A,B,C}(x,y) = \lim_{B\to-\infty} M_{A,B,C}(x,y) = \frac{x\sqrt{|y|} + y\sqrt{|x|}}{\sqrt{|y|} + \sqrt{|x|}} = G_0(x,y)\,,$$

$$\lim_{C\to\infty} M_{A,B,C}(x,y) = \lim_{C\to-\infty} M_{A,B,C}(x,y) = \frac{x+y}{2}\,.$$

37.3 A Higher Dimensional Case

In this section, we prove the following.

Theorem 37.2. *Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 3$, fixed. A strictly monotonic continuously differentiable function $\varphi : I \to \mathbb{R}$ and a continuously differentiable function $f : I \to (0,\infty)$ satisfy the equation $M^{[\varphi]} = M_f$ in I, i.e.*

$$\varphi^{-1}\left(\frac{\sum_{j=1}^{k}\varphi(x_j)}{j}\right) = \frac{\sum_{j=1}^{k} x_j f(x_j)}{\sum_{j=1}^{k} f(x_j)}, \qquad x_1,\dots,x_k \in I, \tag{37.10}$$

if, and only if, either, for some $a,b,c \in \mathbb{R}$, $a \neq 0 \neq c$,

$$\varphi(x) = ax+b, \qquad f(x) = c, \qquad x \in I,$$

and then both means $M^{[\varphi]}$ and M_f are equal to the arithmetic one; or for some $a,b,c,d \in \mathbb{R}$, $a \neq 0 \neq c$,

$$\varphi(x) = \frac{a}{x+d} + b, \qquad f(x) = \frac{c}{x+d}, \qquad x \in I,$$

and then both means $M^{[\varphi]}$ and M_f are equal to the d-translated harmonic one, i.e., for all $x_1,\dots,x_k \in I$,

$$M^{[\varphi]}(x_1,\dots,x_k) = \frac{k}{1/(x+d)+\dots+1/(x+d)} - d = M_f(x_1,\dots,x_k).$$

Proof. For the simplicity of notations assume that $k=3$. Suppose $f : I \to (0,\infty)$ and $\varphi : I \to \mathbb{R}$ are continuously differentiable and satisfy (37.10). Then we have

$$\varphi\left(\frac{xf(x)+yf(y)+zf(z)}{f(x)+f(y)+f(z)}\right) = \frac{\varphi(x)+\varphi(y)+\varphi(z)}{3}, \quad x,y,z \in I.$$

Differentiating this equation, first with respect to x and next with respect to y, gives

$$\varphi'(x) = 3\varphi'\left(M_f(x,y,z)\right) \times \frac{[f(x)+xf'(x)]\,[f(x)+f(y)+f(z)] - f'(x)\,[xf(x)+yf(y)+zf(z)]}{[f(x)f(y)+f(z)]^2},$$

$$\varphi'(y) = 3\varphi'\left(M_f(x,y,z)\right) \times \frac{[f(y)+yf'(y)]\,[f(x)+f(y)+f(z)] - f'(y)\,[xf(x)+yf(y)+zf(z)]}{[f(x)f(y)+f(z)]^2}$$

for all $x,y,z \in I$.

Since φ is strictly monotonic, the set $U := \{x \in I : \varphi'(x) \neq 0\}$ is non-empty and open. Let $J \subset U$ be a nontrivial maximal interval. Since M_f is a mean, we have $M_f(J^3) \subset J$. Dividing these two equations by sides we get

$$\frac{\varphi'(x)}{\varphi'(y)} = \frac{[f(x)+xf'(x)]\,[f(x)+f(y)+f(z)]-f'(x)\,[xf(x)+yf(y)+zf(z)]}{[f(y)+yf'(y)]\,[f(x)+f(y)+f(z)]-f'(y)\,[xf(x)+yf(y)+zf(z)]} \tag{37.11}$$

for all $x,y,z \in J$, which proves that the right hand side does not depend on z. Hence, writing the derivative of the right hand side with respect to z, we get

$$\begin{aligned}
0 = &\left\{\left[f(x)+xf'(x)\right]f'(z)-f'(x)\left[f(z)+zf'(z)\right]\right\}\\
&\times\left\{\left[f(y)+yf'(y)\right][f(x)+f(y)+f(z)]-f'(y)\,[xf(x)+yf(y)+zf(z)]\right\}\\
&-\left\{\left[f(y)+yf'(y)\right]f'(z)-f'(y)\,[f(z)+zf(z)]\right\}\\
&\times\left\{\left[f(x)+xf'(x)\right][f(x)+f(y)+f(z)]-f'(x)\,[xf(x)+yf(y)+zf(z)]\right\}
\end{aligned}$$

for all $x,y,z \in J$. Setting here $y := z$ gives

$$\begin{aligned}
0 = &\left\{\left[f(x)+xf'(x)\right]f'(z)-f'(x)\left[f(z)+zf'(z)\right]\right\}\\
&\times\left\{\left[f(z)+zf'(z)\right][f(x)+2f(z)]-f'(z)\,[xf(x)+2zf(z)]\right\}=0
\end{aligned}$$

for all $x,z \in J$. For $z = x$ the factor in the second brace reduces to $3f(x)^2$. Since, by assumption, f is positive, it follows that for every $x \in J$ there is a non-empty open interval J_x such that $x \in J_x$ and for all $z \in J_x$, the second factor is different than 0. Now the above equation implies that, for every $x \in J$,

$$\left[f(x)+xf'(x)\right]f'(z)-f'(x)\left[f(z)+zf'(z)\right]=0\,, \qquad z \in J_x\,. \tag{37.12}$$

Treating here x as fixed we get

$$\frac{f'(z)}{f(z)} = \frac{\beta}{z+d}\,, \qquad z \in J_x\,,$$

where d and β are some constant (depending on fixed x) and, consequently,

$$f(z) = c(z+d)^{\beta}\,, \qquad z \in I_x\,, \tag{37.13}$$

for some constant $c > 0$. Assume first that $\beta \neq 0$. Setting the function (37.13) into relation (37.12), and performing simple calculations leads to $\beta(z-x) = x-z$ for

$z \in J_x$, whence $\beta = -1$. Now the differentiability of f easily implies that d and c must be constant independent on x. Thus, we have shown that, for some $d \in \mathbb{R}$ and $c > 0$,

$$f(x) = \frac{c}{x+d}, \qquad x \in J. \tag{37.14}$$

If $d = 0$, then

$$M_f(x,y,z) = \frac{xf(x)+yf(y)+zf(z)}{f(x)+f(y)+f(z)} = \frac{3}{1/x+1/y+1/z}, \quad x,y,z \in J,$$

is the harmonic mean. Applying Remark 37.2 we infer that, for some $a, b \in \mathbb{R}$, $a \neq 0$, $\varphi(x) = a/x + b$ for $x \in J$. Now we are going to determine the form of the function φ in the case $d \neq 0$. Setting the function (37.14) into (37.11) and fixing arbitrarily y ,$z \in J$ gives

$$\varphi'(x) = \frac{Ax^2+Bx+C}{Dx^2+Ex+F}, \qquad x \in J, \tag{37.15}$$

for some real constant A,B,C,D,E,F. Replacing the values of φ', f and f' in (37.11) by the suitable expressions given by the formulas (37.14) and (37.15), we obtain an equality of two polynomials of the variables x and y (the variable z disappears). Comparing the coefficients at the relevant monomials leads to the following system of equations for the unknown numbers A,B,C,D,E,F :

$$\begin{gathered}
AF = 0, \qquad AE = 0, \qquad AD = 0, \\
D(2Ad+B) = 0, \qquad E(2Ad+B) = 0, \qquad F(2Ad+B) = 0, \\
AFd^2 + 2BFd + C(F - Dd^2) = 0, \qquad BFd + C(2F - Ed) = 0, \\
AEd^2 + Bd(2E - Dd) + C(E - 2Dd) = 0.
\end{gathered}$$

If $A \neq 0$ then, according to the first three equations of this system, we would have $D = E = F = 0$ which, in view of formula (37.15), is impossible. Thus, $A = 0$ and this algebraic system of equations simplifies to the following one:

$$\begin{gathered}
DB = 0, \qquad EB = 0, \qquad FB = 0, \\
2BFd + C(F - Dd^2) = 0, \qquad BFd + C(2F - Ed) = 0, \\
Bd(2E - Dd) + C(E - 2Dd) = 0.
\end{gathered}$$

Note that $B = 0$ (otherwise we would again have $D = E = F = 0$ which, by (37.15), is impossible). Now the algebraic system of equations implies that $C(F - Dd^2) = 0$, $C(2F - Ed) = 0$, and $C(E - 2Dd) = 0$. According to our assumption, $\varphi' \neq 0$ in J. Since $A = B = 0$, in view of (37.15), we have $C \neq 0$. Therefore $F = Dd^2$ and $E = 2Dd$, whence

$$\varphi'(x) = \frac{C}{Dx^2 + 2D\mathrm{d}x + Dd^2} = \frac{C}{D(x+d)^2}, \qquad x \in J.$$

Hence, making use of the continuous differentiability of φ and the maximality of J we infer that $J = I$, and

$$\varphi(x) = \frac{a}{x+d} + b, \qquad x \in I,$$

where $a := -C/D$ and $b \in \mathbb{R}$. Note that this formula includes also the case $d = 0$.

Assume now that there is an $x \in J$ such that $\beta = 0$ in (37.13). Then the function f is constant in in the interval J_x. Now it is a consequence of the previous reasoning that f is constant in I, i.e. $f(x) = c$ for all $x \in I$. In this case we have

$$M_f(x,y,z) = \frac{xf(x) + yf(y) + zf(z)}{f(x) + f(y) + f(z)} = \frac{x+y+z}{3}, \qquad x,y,z \in I.$$

It follows from (37.9) that, for some $a, b \in \mathbb{R}$, $a \neq 0$, $\varphi(x) = ax + b$ for $x \in I$. The verification of the converse implication is a matter of easy calculations. □

Remark 37.8. Let us mention that the problem of equality of k-variable weighted quasi-arithmetic means of the form

$$\varphi^{-1}\left(\frac{\sum_{j=1}^{k} g(x_j)\varphi(x_j)}{g(x_j)}\right)$$

with $k \geq 3$, under the assumptions that the involved functions are twice continuously differentiable, was considered by Bajraktarević [1].

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